Volume VII. *July, 1898.* *Number 1.*

THE

PHYSICAL REVIEW.

ON THE RELATION BETWEEN PRESSURE, CURRENT, AND LUMINOSITY OF THE SPECTRA OF PURE GASES IN VACUUM TUBES.

ERVIN S. FERRY.

ON acount of our almost entire dependence upon the use of the spectroscope in the investigation of the history and present condition of the heavenly bodies, as well as for the study of the ultimate constitution of matter, a knowledge of the changes produced in spectra by different circumstances is of the greatest theoretical importance. The application, also, of spectrophotometry to the measurement of high temperatures and the determination of the quantitative composition of compounds, gives the subject an added interest from a practical point of view.

Bolometric measurements made by K. Ångström [1] show that the radiation of a gas rendered luminous by electricity is proportional to the current strength—at least within the wide limits of his experiments. He also showed that this proportionality not only holds for the total radiation, but also for the luminous part of the radiant energy. From this, we would expect that the same law would obtain between the current strength and the luminosity of the separate spectral lines. Results obtained by Lagarde [2] on the spectrum of hydrogen appear to oppose this view. Lagarde con-

[1] Nova Acta Reg. Soc. Ups. III., 1892.

[2] Recherches Photométrique sur la spectre de l' hydrogéne—Am. de Chem. et de Phys. VI., 4, p. 248 (1885).

cludes that under constant pressure the luminosity of a line of the spectrum of hydrogen varies as the logarithm of the current strength, where the logarithmic base is nearly equal to unity. But it must be noticed that Lagarde used a Ruhmkorff coil to make the gas luminous, and as E. Wiedemann[1] had already pointed out, the complicated train of partial discharges following each principal discharge of the secondary coil, produces composite spectral lines whose luminosity is influenced by each of these partial discharges. And the amount and distribution of energy in these trains of discharges is greatly affected by several such circumstances as the action of the interrupter in the primary circuit and the resistance of the discharge tube. Again, the determination of the current in such a discharge is a difficult measurement to perform, and the employment of a galvanometer for this measurement in the manner adopted by Lagarde could not be expected to give an accuracy comparable with that obtainable in the measurement of the other quantities of the experiment. In view of our uncertainty regarding this most important question it has been thought that a more detailed study would be of value.

The object of the present investigation has been to study photometrically the changes produced in the spectra of pure gases when subjected to various conditions of current and pressure. And being supplementary to Ångström's "Bolometrische Untersuchung über die Stärke der Strahlung verdünnter Gase unter dem Einflusse der elektrischen Entladung," the apparatus and methods employed in his investigation were also employed in this, with the exception of the changes made necessary by the substitution of a spectro-photometer for a bolometer.

The discharge tube was kept in permanent connection with a mercury pump supplied with a MacLeod manometer so that the pressure could be altered and measured at will. Intercalated between the pump and discharge tube was a barometer tube[2] for the introduction of the gases to be examined, and tubes filled with phosphoric acid anhydride, sulphur, and copper for the removal of water

[1] Ueber das thermische und optische Verhalten von Gasen unter dem Einflusse der electrischen Entladung, Pogg. Am. X., p. 202 (1880).

[2] A. Cornu, Journal de Physique, V., p. 100 (1886).

vapor, mercury vapor, and sulphur vapors respectively. There were no stop cocks in the apparatus and all joints were made by fusing the parts together.

Before the apparatus was made, each piece of glass tubing was cleaned in succession with alcohol, nitric acid, and distilled water, and after the blow-pipe work had been done the finished parts were again cleaned in the same manner and thoroughly dried by drawing filtered dry air through them. The discharge tubes were treated in addition as follows: they were partly filled with distilled water and purified sand and allowed to rotate for about six hours on a whirling table in the hope of removing by attrition any solid carbon compounds that might have been introduced by the blow-pipe flame. They were then rinsed with distilled water and then with a current of steam from a glass distilling apparatus. After the apparatus had been assembled, the whole system of tubes was repeatedly rinsed with pure dried electrolytically prepared hydrogen, this thoroughly pumped out and then replaced by oxygen. When the apparatus was filled with oxygen an electric discharge was passed through the whole apparatus by connecting the secondary terminals of an induction coil to pieces of tin-foil wrapped around sections of the tubing about half a meter apart. This produces ozone, which has a powerful oxidizing effect upon impurities that have resisted the previous treatment. In some cases the reaction was assisted by heating the evacuated tubes to a dull red heat with a Bunsen burner; but as the increased purity of the tubes obtained from this treatment did not appear commensurate with the danger of the tubes cracking where the electrodes entered, this method was resorted to in but a few instances.

In order to prevent the materials of the glass affecting the spectra, the discharge tubes employed were of about one centimeter bore. They were supplied with aluminium electrodes for the entrance of the exciting current and small platinum terminals connected to a Mascart quadrant electrometer for the measurement of the potential difference existing between two fixed points about five centimeters apart. An accumulator of twelve hundred elements made by Klingelfuss, of Basel, was employed to render luminous the gas in the discharge tube. A d'Arsonval galvanometer and a resistance

formed by a ten per cent. solution of cadmium iodide in amyl alcohol allowed the amount of current passing through the tube to be varied in known amounts. The spectroscope was furnished with two prisms and a Vierordt double slit adjustable by two fine micrometer screws. One slit looked along the axis of the discharge tube and the other received the light from the comparison lamp by means of a silvered glass mirror placed at 45° to the axis of the collimator. The position of the various parts of the apparatus was so adjusted that the spectra from the gas and that from the comparison lamp were formed side by side at the focus of the observing telescope with a scarcely perceptible line of separation between them. The light used to produce the comparison spectrum was furnished by an argand gas burner in connection with a pressure regulator, which, as shown by a previous experiment, kept the pressure of the gas constant to within two per cent. for long periods. In the focal plane of the telescope was a diaphragm containing a slit that could be adjusted in width to suit the spectral line whose luminosity was under observation.

The method of observation consisted in first pumping out the apparatus until the pressure was about 0.01 mm. of mercury, then introducing through the barometer tube bubbles of the gas being studied until the pressure was about 4.0 mm. The luminosity of selected lines of the spectrum was then measured in terms of the standard spectrum from the comparison light source, for different current strengths from one milliampère to six milliampères. The potential difference existing between the electrometer terminals in the discharge tube was also observed. A little mercury was then allowed to flow through the pump, the reduced gas pressure measured with the MacLeod manometer, and the luminosity of the same spectral lines again determined for various current strengths. Similar sets of observations were taken at different decreasing pressures to about 0.25 mm.

In this manner were examined the line spectrum of hydrogen and the band spectrum of nitrogen. The apparatus was also put together once with the omission of the sulphur and copper tube so that the mercury vapor from the pump freely entered the discharge tube and produced its brilliant spectrum superposed on the nitrogen spectrum.

The luminosity of the green mercury line was then measured for different current strengths. Again, through lack of the excessive pains necessary to eliminate impurities from the apparatus, opportunities were afforded, only too often, for the study of the so-called "second hydrogen spectrum." It is comparatively easy to obtain spectra of nitrogen, oxygen, and many other gases in which this spectrum does not appear even when the exhaustion of the discharge tube is carried to the lowest degrees; but with some other gases this "second hydrogen spectrum" comes out brilliant when the exhaustion is carried below one or two millimeters of mercury.

The hydrogen was prepared electrolytically from recently boiled distilled water acidulated with phosphoric acid anhydride. The nitrogen was produced by the action of sodium nitrite on ammonium chloride purified by passing through a long, nearly horizontal tube filled with strong sulphuric acid.

On account of the lack of sensitiveness of the eye to changes in the luminosity of light of greater refrangibilities, lines from the red to green part of the spectrum only were selected for photometry. Also, on account of the difficulty of accurately comparing the luminosity of two spectra differing greatly in purity, due care was taken in the adjustment of the effective intensity of the comparison light, and in the selection of the spectral lines to be studied, to prevent the necessity of having the two slits of the spectrophotometer differ greatly in width at the moment when the images of the two sources were of the same luminosity at the eye piece of the observing telescope.

One set of observations on hydrogen line $\lambda = 6563$ is given in Table I. The luminosities are given in scale divisions of the micrometer screw attached to the slit through which passes the light from the comparison lamp. The other slit, *i. e.*, the one opposite the discharge tube, is kept of constant width. The current strengths used were: one, two, four, and six milliampères. A greater current than this last amount would melt the electrodes. The values given in this table are shown graphically in Fig. 1, where the abscissas represent pressures, and the ordinates luminosities. Four curves are given, for current strengths of one, two, four, and six milliampères.

TABLE I.

LUMINOSITY OF HYDROGEN LINE $\lambda = 6563$ FOR DIFFERENT CURRENTS AND PRESSURES.

Pressure of gas in mm. of Hg.	3.8	2.7	1.4	0.7
Electrometer readings	140	110	80	65
Luminosity—current = 1 m.a.	2	3	4	5
" " = 2 "	4	6	8	10
" " = 4 "	8	12	16	20
" " = 6 "	13	18	24	29

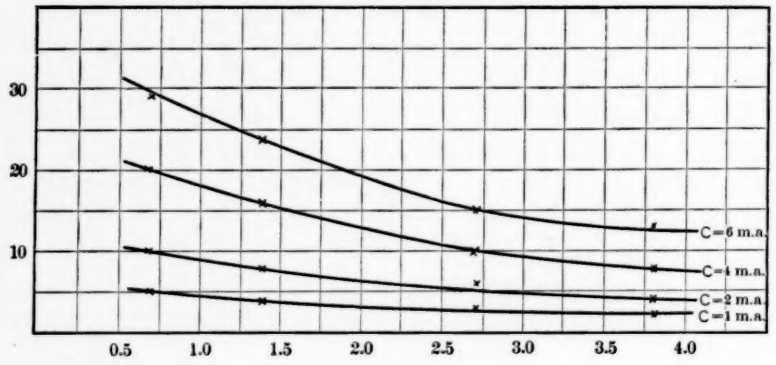

Fig. 1.

Table II. gives similar readings for nitrogen bands $\lambda = 6701$, 6622, 6542, and 6465, while Fig. 2 shows graphically the relations between luminosity, pressure and current for bands 6701 and 6542.

TABLE II.

LUMINOSITY OF NITROGEN BANDS FOR DIFFERENT CURRENTS AND PRESSURES.

Nitrogen band $\lambda = 6701$.					
Pressure of gas in mm. of Hg.	4.4	2.2	1.3	0.7	0.3
Electrometer readings	300 mm.	200 mm.	120 mm.	100 mm.	50 mm.
Luminosity—current = 1 ma.	22	24	26	30	35
" " = 2 "	44	49	53	61	68
" " = 3 "	66	73	79	93	105
" " = 4 "	88				
" " = 6 "	133	145	160	178	215
Nitrogen band $\lambda = 6622$.					
Luminosity—current = 1 ma. —	23	25	28	33	43
" " = 2 " —	47	48	56	65	83
" " = 3 " —	75	75	83	97	128
" " = 4 " —	100				
" " = 6 " —	145	148	168	193	257

Nitrogen band $\lambda = 6542$.

Pressure of gas in mm. of Hg.	4.4	2.2	1.3	0.7	0.3
Luminosity—current = 1 ma. —	13	15	18	22	26
" " = 2 " —	26	29	35	45	51
" " = 3 " —	40	45	53	66	78
" " = 4 " —	53				
" " = 6 " —	78	88	108	133	158

Nitrogen band $\lambda = 6465$.

Luminosity—current = 1 ma. —	8	10	12	14	18
" " = 2 " —	17	20	25	27	35
" " = 3 " —	27	31	36	43	54
" " = 4 " —	36				
" " = 6 " —	54	61	73	83	109

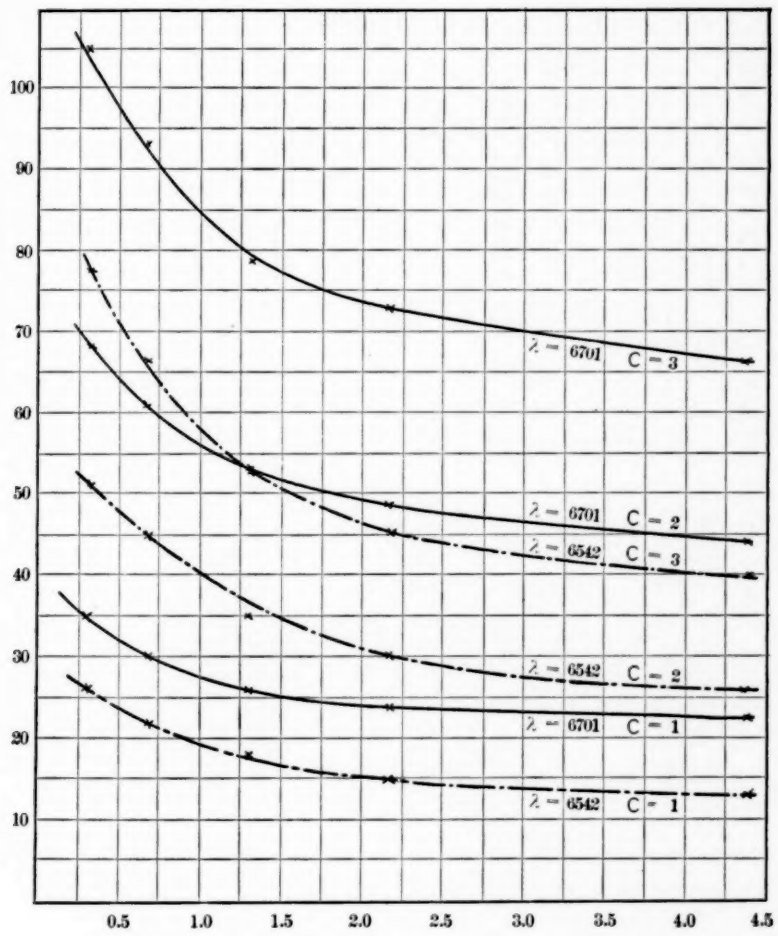

Fig. 2.

In the same manner Table III. and Fig. 3 give observations in "second hydrogen spectrum" of $\lambda = 6198$, 6069, 6052, and 5735.

TABLE III.

LUMINOSITY OF BANDS OF THE "SECOND HYDROGEN SPECTRUM" FOR DIFFERENT CURRENTS AND PRESSURES.

"Carbon spectrum" band $\lambda = 6198$.						
Pressure of gas in mm. of Hg.	4.25	2.75	1.85	1.1	0.65	0.275
Electrometer readings	60	55	48	35	25	15
Luminosity—current = 2 ma.	2	3	4	5	8	11
" " = 4 "	4	7	8	10	15	23
" " = 6 "	6	10	12	15	24	
"Second hydrogen spectrum" band $\lambda = 6069$.						
Luminosity—current = 2 ma.	3	4	5	6	9	13
" " = 4 "	7	9	10	13	18	26
" " = 6 "	9	12	16	21	27	
"Second hydrogen spectrum" band $\lambda = 6052$.						
Luminosity—current = 2 ma.	3	3	6	7	9	14
" " = 4 "	7	8	10	14	19	27
" " = 6 "	9	12	17	21	27	
"Second hydrogen spectrum" band $\lambda = 5735$.						
Luminosity—current = 2 ma.	3	4	5	7	9	13
" " = 4 "	6	9	10	14	18	27
" " = 6 "	9	12	16	21	27	

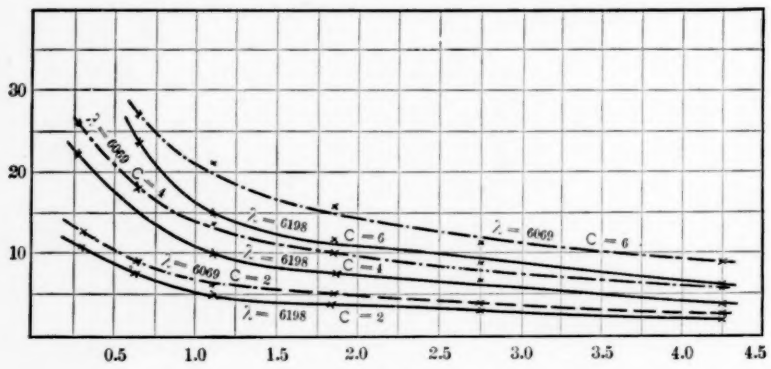

Fig. 3.

Since the arrangement of the apparatus did not permit the determination of the vapor tension of the mercury vapor, a set of tables for various and determined pressures could not be obtained. But values were taken on several occasions of the luminosity of the

bright green line at λ 5460 for various current strengths, a typical set of which is given in Table IV. The tube was filled with nitrogen, and mercury was the only visible impurity. The pressure of the gas mixture as determined from the measurement of the height of mercury in a barometer tube was about 0.6 mm.

TABLE IV.

Luminosity of mercury line $\lambda = 5460$.

Current = 1 milliampère — luminosity = 18
Current = 2 milliampère — luminosity = 36
Current = 3 milliampère — luminosity = 55
Current = 4 milliampère — luminosity = 73
Current = 5 milliampére — luminosity = 90

A survey of the preceding tables and curves leads to the following conclusions regarding the luminosity of the separate spectral lines of gases in vacuum tubes under different conditions of pressure and current strength :

I. With gas pressure of from 0.25 mm. to 4.00 mm. of mercury, and current strengths from one milliampère to six milliampères, the luminosity of the separate spectral lines of gases at a given pressure is directly proportional to the current strength. This is in exact parallelism with K. Ångström's observation that the energy of the luminous radiation as well as the total radiant energy of a gas rendered luminous by an electric current, is directly proportional to the current strength.

II. With constant current, the luminosity of a spectral line of a gas increases as the pressure decreases, at first slowly and than more rapidly. The curve showing the relation between the pressure of the gas and the luminosity of a spectral line is regular, but is different for different lines.

This research was performed in the Physical Institute of the University at Upsala, and in conclusion I wish to express my deep obligations and grateful thanks to Professor Kunt Ångström for the assistance and encouragement I have received from him throughout the entire course of these experiments.

BALLISTIC GALVANOMETRY WITH A COUNTER-TWISTED TORSION SYSTEM.

BY C. BARUS.

1. The ballistic galvanometer is ordinarily used in one of two ways: the needle may be quite undamped; in this case the charge sent through the galvanometer is proportional to the first deflection, and is given absolutely if the horizontal intensity of terrestrial magnetism, the galvanometer constant, and the period of the needle are known, provided the charge subsides completely in a time negligible with the period in question. In the second case the needle is damped, and the logarithmic decrement corresponding to the degree of damping applied is suitably predetermined. With this datum, in addition to those mentioned for the unhampered magnetic system under the conditions stated, the instrument again admits of an absolute measurement by the charge flowing through the galvanometer, though the experiment is naturally one of greater difficulty.

2. I propose to use the ballistic galvanometer in a way somewhat different from this, which will, I think, offer some advantages. The plan is to provide the needle with *stops* so that it can be deflected in one direction only (say clockwise), and to hold it against these stops by the torque of a suitable torsion fiber twisted counter-clockwise. If now the charge Q is whisked through the galvanometer, the (clockwise) deflection of the needle will be definitely less, in proportion as the applied counter torsion is greater, *caet. par.* Now let the applied twist, θ_1, be such as to reduce the deflection to a small arc, θ_2, of fixed value in all the experiments. Then the charge must be directly expressible in terms of the observed twist (θ_1) of the fiber and the constants of the instrument of which the small arc, θ_2, is now one.

When used in this way the needle practically does not leave the plane of the coils. The small deflection, θ_2, is determined by a flash

of light, while θ_1 the twist of the fiber is read off on a torsion circle. In virtue of the stopped needle and counter-torsion, greater stability has been given to the galvanometer system. An instrument with an *undamped* needle may be constructed in portable form, ready for immediate use. The earth's magnetism may be eliminated in favor of torsion, either by an astatic needle, or from a different point of view by the D'Arsonval method. A simple mechanical device at the stops enables the observer to apply twists in opposite directions. The flash may be replaced by an electrical contact registry, etc.

3. To enter into the theory of the instrument it is advisable to begin with the general case of a damped needle. For brevity I will suppose this to be astatic and determine the corrections due to terrestrial magnetism (effective throughout the small arc, θ_2, only) later. So understood the needle is actuated by the torsion of the fiber, the damping resistance, and the deflecting torque of the transition current, $\dot{Q}$, to be considered.

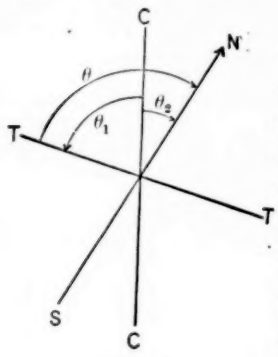

Fig. 1.

In the let figure CC be the plane of the coils, and therefore the equilibrium position as well as the stopped position of the needle, NS, and the initial or null position of the torsion index TT. Suppose that at the time, t, while the twist θ_1 has been permanently imparted to the fiber, a current whose instantaneous value is $\dot{Q}$ is passing, and the needle, NS, shows a relatively very small deflection θ_2. The total torque on the needle due to the fiber is therefore $C(\theta_1 + \theta_2) = C\theta$, where C is the modulus of torsion.

Let $\mathfrak{M}$ be the magnetic moment of the needle, N its moment of inertia, G the galvanometer constant; and let k be the damping coefficient, so that $k\dot{\theta}$ denotes the resisting torque from this cause.

The equation of motion may now be written

$$\ddot{\theta} + \frac{k}{N}\dot{\theta} + \frac{C}{N}\theta = \frac{\mathfrak{M}G}{N}\dot{Q} \tag{1}$$

This is a familiar differential equation, occurring as it does in the theory of resonance in acoustics, in the treatment of alternating cur-

rents, and elsewhere. The full integration is conditioned by the occurrence of $\dot{Q}$; otherwise Lagrange's method of varying the constants reduces the expression to its simplest form.

For brevity let

$$(2) \qquad a = \frac{k}{2N}, \quad b = \frac{\sqrt{4CN - k^2}}{2N}, \quad A = \frac{\mathfrak{M}G}{Nb}.$$

Then the integral may be written

$$(3) \qquad \theta = \varepsilon^{-at}(c_1 \cos bt + c_2 \sin bt),$$

where c_1 and c_2 are the variable parameters to be determined. It suffices to differentiate (3) twice with respect to t, regarding c_1 and c_2 as time functions in the usual way, and to reduce the result by (1). This gives

$$(4) \qquad \dot{C}_1 = -A\,\varepsilon^{at}\,\dot{Q}\sin bt$$

$$(5) \qquad \dot{C}_2 = A\,\varepsilon^{at}\,\dot{Q}\cos bt$$

Integrating and replacing A, a, b, by their values the results are finally

$$(6) \qquad c_1 = -\frac{2\mathfrak{M}G}{\sqrt{4CN-k^2}}\int \varepsilon^{\frac{kt}{2N}}\,\dot{Q}\sin\frac{\sqrt{4CN-k^2}}{2N}\,t\,dt + c$$

$$(7) \qquad c_2 = \frac{2\mathfrak{M}G}{\sqrt{4CN-k^2}}\int \varepsilon^{\frac{kt}{2N}}\,\dot{Q}\cos\frac{\sqrt{4CN-k^2}}{2N}\,t\,dt + c'$$

$$(8) \qquad \theta = \varepsilon^{-\frac{kt}{2N}}\left(c_1\cos\frac{\sqrt{4NC-k^2}}{2N}\,t + c_2\sin\frac{\sqrt{4CN-k^2}}{2N}\,t\right)$$

Let τ be the time which elapses from the instant the needle is lifted off the stops to the instant of maximum elongation. The point next at issue is then the determination of the arbitrary constants c and c' in terms of θ_1 θ_2 and τ and the constants C, N, k of the instrument.

Since the needle has some freedom of motion its period and logarithmic decrement may be found by observation. Hence the period of the undamped needle may be computed in the usual way, so that $\sqrt{N/C}$ may be considered known.

The meaning of c is obvious from (3) or (8), and (6). If $t = 0$, $\theta = c_1 = c = \theta_1$. Thus $c = \theta_1$ is the twist imparted to the torsion fiber to produce the initial counter-torque $C\theta_1$.

To find c', (3) is to be differentiated recalling that when $t = \tau$, $\dot{\theta} = 0$ and $\theta - \theta_1 = \theta_2$. Hence $c = \theta_1$,

$$c' = \frac{a + b\tan b\tau}{b - a\tan b\tau}\left(\theta_1 - A\int_0^\tau \varepsilon^{at}\,\dot{Q}\sin bt\cdot dt\right) - A\int_0^\tau \varepsilon^{a\tau}\,\dot{Q}\cos bt\cdot dt, \tag{9}$$

which for brevity embodies the assumption that $\dot{Q} = 0$ when $t = 0$. In the exceptional case where a steady current runs through the galvanometer initially, the notation of (9) would require modification.

We may note in passing, however, that the initial torsion may in many cases be supplemented by the torque of a steady current and the throw observed on breaking.

Equations (8) and (9) together make up the complete integral of (1) required, but this form of θ, call it (10) when written out in full is too cumbersome to need reproduction here.

If in the value of θ so found we make $t = \tau$, and, therefore, $\theta = \theta_1 + \theta_2$, equation (10) reduces to

$$(\theta_1 + \theta_2)\,\varepsilon^{a\tau} = b\left(\theta_1 - A\int_0^\tau \varepsilon^{at}\,\dot{Q}\sin bt\cdot dt\right)\frac{\sqrt{1 + \tan^2 b\tau}}{b - a\tan b\tau} \tag{11}$$

To integrate this equation $\dot{Q}$ must be given as to the form of its variation relative to time.

4. Among the special forms of $\dot{Q}$ the most important is that in which $\dot{Q}$ completely subsides before $\sin bt$ or bt becomes significant. In this case one factor of the integrant in (11) will always be appreciably zero, and since $\dot{Q}$ must remain finite the integral vanishes. Hence

$$\frac{\theta_1 + \theta_2}{\theta_1} = \frac{be^{-a\tau}}{b\cos b\tau - a\sin b\tau}, \tag{12}$$

if the current quite passes before the needle has sensibly moved.

Put (13)
$$R = \frac{(\theta_1 + \theta_2)\,\varepsilon^{a\tau}}{\theta_1}.$$

Then these results successively follow

$$\sin b\tau = -\frac{ab}{a^2+b^2}\frac{1}{R} + \frac{b}{\sqrt{a^2+b^2}}\sqrt{1-\frac{b^2}{a^2+b^2}\frac{1}{R^2}}, \tag{14}$$

$$\cos b\tau = \frac{b^2}{a^2+b^2}\frac{1}{R} + \frac{a}{\sqrt{a^2+b^2}}\sqrt{1-\frac{b^2}{a^2+b^2}\frac{1}{R^2}}, \tag{15}$$

$$c' = \theta_1\sqrt{\frac{a^2+b^2}{b^2}R^2-1} = \theta_1\sqrt{\frac{2C}{4CN-k^2}R^2-1}. \tag{16}$$

The present value of c and c' inserted in equation (10) reduce it to the simplified form

$$\theta = \varepsilon^{-at}\left(\cos bt + \frac{a+b\tan b\tau}{b-a\tan b\tau}\sin bt\right)\theta_1 \tag{17}$$

5. It seems curious that $\dot{Q}$ is thus wholly removed from (17), though it would not otherwise have been possible to obviate the integration in (11). $\dot{Q}$ is implicitly contained in θ, however, since $[d\theta/dt]^2$ for $t=0$ is proportional to the initial energy of the needle, and, therefore, proportional to the square of the charge which has passed instantaneously through the instrument. It will thus be necessary to take this roundabout step to restore Q to the equation.

Instead of taking (17), it is preferable, for reasons which appear below, to differentiate the fundamental equation (3). Thus

$$\dot{\theta}+a\theta = b\varepsilon^{at}\left\{\left(-\sin bt\left(c - A\int\varepsilon^{at}Q\sin bt\cdot dt\right.\right.\right.$$
$$\left.\left.\left.+\cos bt\left(c'+A\int\varepsilon^{at}\dot{Q}\cos bt\cdot dt\right)\right)\right\} \tag{18}$$

when c and c' are given in (9). Now if t becomes indefinitely small without quite reaching zero, $\dot{\theta}=\dot{\theta}_0$, the initial angular velocity, and $\theta=\theta_1$ in the limit. The first integral of (18) vanishes and the second is AQ. This however must also vanish except for the case of the present paragraph where $\dot{Q}$ in an initial impulse. Hence for vanishing time and instantaneous $\dot{Q}$

$$\dot{\theta}_0 = -a\theta_1 + bc' + bAQ. \tag{19}$$

Substituting (9) for the case of impulsive $\dot{Q}$

$$\dot{\theta}_0 = \theta_1 \frac{(a^2 + b^2)\tan b\tau}{b - a\tan b\tau}, \tag{20}$$

An equation which follows at once from (17).

The initial energy $\frac{1}{2} N\dot{\theta}_0^2$ of the needle being now given in terms of (20), may be replaced by its electrical equivalent $\frac{1}{2}(\mathfrak{M}GQ)^2/N$. Thus

$$\mathfrak{M}GQ = N\theta_1 \frac{(a^2 + b^2)\tan b\tau}{b - a\tan b\tau} = CN\theta_1 \frac{2\tan b\tau}{\sqrt{4CN - k^2} - k\tan b\tau} \tag{22}$$

Finally $\tan b\tau$ found from (14) and (15) or otherwise is

$$\tan b\tau = \frac{\sqrt{4CN(1 - k^2/4CN)}}{1 - (R^2 + 1)k^2/(4CN)^2}\left(-\frac{k}{4CN}R^2 + \sqrt{\frac{k^2}{(4CN)^2} + \frac{R^2 - 1}{4CN}}\right), \tag{23}$$

where $R = (\theta_1 + \theta_2)\, e^{k\tau/2N}/\theta_1$. If the damping is not excessive, $a = k/2N$ and usually $k/4CN$ will be small quantities. Hence $\tan b\tau$ may be computed with any degree of approximation desired.

6. Now it is precisely the undamped needle which is of interest in the present paper, the idea being to secure the convenience of damping by a suitably stopped but undamped needle. The purpose of introducing $a = k/2N$ has been to ascertain the character of the corrections in this case rather than to find the working equation. This is at once obtained by making $a = k = 0$ in equations (22) and (23), when

$$Q = \frac{\sqrt{2CN\cdot\theta_2}}{\mathfrak{M}G}\sqrt{\theta_1}, \tag{24}$$

the small constant reflection θ_2 being regarded as one of the constants of the instrument. Thus the change passing suddenly through the galvanometer is proportional to the square root of the twist θ_1 imparted to the fiber. The needle is thrown out from the steps to an angular distance θ_2 and then falls back again.

In this case moreover many of the angular quantities which appear under involved expressions in § 4, take very simple forms.

Equation (15) now becomes

$$\cos b\tau = \cos\sqrt{\frac{C}{N}}\cdot\tau = \theta_1/(\theta_1 + \theta_2). \tag{25}$$

If θ_2 and therefore $\tau \cdot \sqrt{C/N}$ is small, the time τ which elapses between the instant of leaving the stops and of maximum elongation eventually because

$$\tau = \sqrt{\frac{2N}{C}\frac{\theta_2}{\theta_1 + \theta_2}}$$

or it decreases nearly as $\sqrt{\theta_2/\theta_1}$.

7. Equation (24) might have been found directly. The work done in twisting the fiber from θ_1 to θ_2 is by a simple integration volving Hooke's law, $\frac{1}{2}C\theta_2(\theta_1 + \theta_2)$ or nearly $C\theta_1\theta_2$. The energy supplied by the transient current is as above $(G\mathfrak{M}Q)^2/2N$. Equating these values reproduces (24).

This deduction may be made to include the earth's magnetic field, if this is not negligible. The potential energy of the magnet is—$\mathfrak{M}\mathfrak{H}$ (cos $\mathfrak{M}$,$\mathfrak{H}$) if $\mathfrak{H}$ is the earth's horizontal magnetic intensity and $\mathfrak{M}$ the effective magnetic moment. For the small deflection θ_2, the work done against the field is $\mathfrak{M}\mathfrak{H}(1 - \cos\theta_2) = 2\mathfrak{M}\mathfrak{H}\sin^2\theta_2/2 = \mathfrak{M}\mathfrak{H}\theta^2/2$. Thus

$$(G\mathfrak{M}Q)^2/2N = C\theta_1\theta_2 + {}^1(C + \mathfrak{M}\mathfrak{H})\theta_2^2$$

or

$$(26) \qquad Q = \frac{\sqrt{2CN\theta_1\theta_2}}{\mathfrak{M}G}\sqrt{1 + \frac{\theta_2}{2\theta_1}\left(1 + \frac{\mathfrak{M}\mathfrak{H}}{C}\right)}.$$

The first factor in (26) is identical with (24). The other factor is the magnetic correction which vanishes with θ_2/θ_1. This equation therefore is under conditions of small θ_2 applicable to the D'Arsonval galvanometer in which case $\mathfrak{M}$ would be the effective magnetic moment of the coils, $\mathfrak{H}$ the intensity of the artificial field.

8. To introduce the effect of a magnetic field in case of a damped needle, the original equation (1) is to be modified by the addition of the magnetic torque. This for the deflection θ is $-\mathfrak{M}\mathfrak{H}(\theta - \theta_1)$, for θ_2 the maximum value of $\theta - \theta_1$, is small. Hence (1) becomes

$$(27) \qquad \ddot{\theta} + \frac{k}{N}\dot{\theta} + \frac{C + \mathfrak{H}\mathfrak{M}}{N}\theta = \mathfrak{M}\frac{G\dot{Q} + \mathfrak{H}\theta_1}{N}$$

To allow for the magnetic effect, therefore, C in the integrals (6), (7), (8), etc., is to be replaced by $C' = C + \mathfrak{M}\mathfrak{H}$, while simulta-

neously $G\dot{Q}$ is replaced by $G\dot{Q} + \mathfrak{H}\theta_1$. The former substitution occasions no difficulty but the latter introduces two new integrals into (10), or one each into (6) and (7). The value of these integrals, apart from the constant factor

$$\frac{2\mathfrak{H}\mathfrak{M}\theta_1}{\sqrt{4CN - k^2}},$$

common to both, is

$$-\int \varepsilon^{at} \sin bt dt = -\varepsilon^{at} \frac{a \sin bt - b \cos bt}{a^2 + b^2},$$

and

$$\int \varepsilon^{at} \cos bt\, dt = \varepsilon^{at} \frac{a \cos bt + b \sin bt}{a^2 + b^2}.$$

The effect of this, seeing that the new terms are multiplied repectively by cos bt and sin bt, is to add the expression

$$\frac{\mathfrak{M}\mathfrak{H}\theta_1}{C'} = \frac{\mathfrak{M}\mathfrak{H}\theta_1}{C + \mathfrak{M}\mathfrak{H}} \tag{28}$$

to the right hand member of (8) or (10). The equation equivalent to (10) thus is

$$\theta = \varepsilon^{-at} \left\{ \cos b't \left(c'' - A \int \varepsilon^{at} \dot{Q} \sin b't \cdot dt \right) + \sin b't \left(c''' + A \int \varepsilon^{at} \dot{Q} \cos b't dt \right) \right\} + \frac{\mathfrak{M}\mathfrak{H}\theta_1'}{a^2 + b'^2}. \tag{29}$$

where the quantities θ', b', θ_1', c'', c''' are accentuated to denote that they refer to $C' = C + \mathfrak{M}\mathfrak{H}$ and the new magnetic conditions. To grasp the meaning of the new equation put

$$\theta' = \theta_1' + \theta_2'$$

where θ_1' is the angle of twist and θ_2' the throw of the needle whose maximum value is θ_2. Then if the constant term of twenty-nine be placed in the left hand member, this becomes on reduction,

$$\theta_1' (1 - \mathfrak{M}\mathfrak{H}/C') + \theta_2'.$$

In other words all permanent twists θ_1' along which the magnetic torque does not act are to be reduced in the ratio of $1 / (1 - \mathfrak{M}\mathfrak{H}/C')$. It is therefore easy to write out the present values of the quantities

given in the above equations. Thus in the presence of an external magnetic field

$$c'' = \theta_1' \left(1 - \mathfrak{M}\mathfrak{H} / C'\right) \tag{30}$$

$$c''' = \frac{a + b\tan b'\tau}{b - a\tan b'\tau}\left\{\theta_1'\left(1 - \mathfrak{M}\mathfrak{H}/C'\right) - A\int e^{a\tau}\dot{Q}\sin b'\tau d\tau\right) - A\int e^{a\tau}\dot{Q}\cos b'\tau d\tau\Big)\Big\} \tag{31}$$

$$\dot{\theta}' = -\left(\theta_1'\left(1 - \mathfrak{M}\mathfrak{H}/C'\right) + \theta_2'\right) + b'\varepsilon^{-at}\left\{-\sin b't\left(c'' - A\int e^{at}\dot{Q}\sin b'tdt\right) + \cos b't\left(c''' + A\int e^{at}\dot{Q}\cos b'tdt\right)\right\} \tag{32}$$

$$\dot{\theta}_0' = \theta_1'\left(1 - \mathfrak{M}\mathfrak{H}/C'\right)\frac{(a^2 + b^2)\tan b'\tau}{b - a\tan b'\tau} - \frac{a + b\tan b'\tau}{b - a\tan b'\tau}\,b'A\int e^{at}\dot{Q}\sin b\tau d\tau \tag{33}$$

$$\left(\theta_1'\left(1 - \mathfrak{M}\mathfrak{H}/C'\right) + \theta_2\right)e^{at} = \frac{b'}{b'\cos b'\tau - a\sin b'\tau}\left(\theta_1'\left(1 - \mathfrak{M}\mathfrak{H}/C'\right) - A\int e^{at}\dot{Q}\sin b'\tau d\tau\right) \tag{34}$$

$$R = \left(\theta_1'\left(1 - \mathfrak{M}\mathfrak{H}/C'\right) + \theta_2\right)e^{a\tau}/\theta_1'\left(1 - \mathfrak{M}\mathfrak{H}/C'\right). \tag{35}$$

In all these equations the form of $\dot{Q}$ is arbitrary. If $\dot{Q}$ is an initial impulse the sine integral vanishes and the cosine integral becomes AQ, with a corresponding simplification of the equations. Finally if $a = k = 0$, the value of R in (35) substituted in (23) above, and this in (33), reproduces the working equation (26) already given, since $\mathfrak{M}GQ = N\dot{\theta}_0$ the initial angular momentum.

Tentative experiments made by Mr. A. E. Watson of our laboratory by aid of this equation gave good results even for a moderately damped needle. I have not myself as yet studied the instrument experimentally, and for this reason various interesting combinations which the above equations present are left without comment.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

AN APPLICATION OF INTERFERENCE METHODS TO A STUDY OF THE CHANGES PRODUCED IN METALS BY MAGNETIZATION.

By J. S. Stevens.

PROBLEMS which have for their object a study of the changes produced in metals when magnetized have, during the past fifty years, received attention from physicists. It is thought that if a thorough investigation of the subject could be made, much light would be thrown upon the phenomenon of magnetization, and even the theory of the constitution of matter. Historical notes relating to this problem may be found in the Physical Review, Vol. III., No. 15, and Vol. V., No. 5. The results of various investigators agree fairly well as to the maximum elongation of a rod or wire when saturated. There seems, however, to remain some difference of opinion concerning the behavior of rods under states of magnetization below the saturation point. If the general proposition that the change depends largely upon the kind and condition of the metal used may be assumed as correct, these discrepancies may be accounted for.

The important investigations made by Professor Rogers and others, in which interference methods were applied to the measurement of small temperature changes in the length of rods, suggested to me the possibility of applying similar methods to a study of this problem. The "interferometer" has now become so well known that it will not be necessary to describe it here. This instrument was used with slight modification of its construction. The mirror frame, which is commonly attached to the movable bed, was removed and firmly screwed to the end of the rod to be tested (A in the figure). The other end of the rod was made fast to a brass cylinder (K in the figure), which could be moved backwards and forwards, and thus provide the adjustment usually afforded when the mirror is moved by means of a micrometer screw. The magnetizing coil was

supported at *S* and the support as well as the interferometer were clamped in a slot so that adjustment in two directions was possible. The whole was attached to a heavy wooden frame, and little or no inconvenience was caused by vibrations in the building.

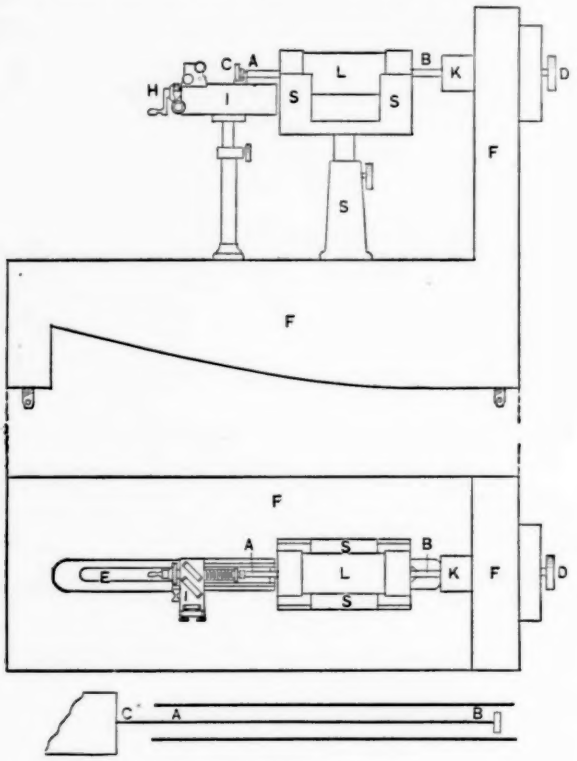

Fig. 1.

In the first part of the experiment, when it was desired principally to find the best working conditions and make some relative measurements, the precaution was not taken to make the coil longer than the rods, and the rods, while equal in diameter, were too large to afford the most satisfactory results. The coil was composed of 15 layers of no. 25 copper wire and was 21 cm. long. It had a resistance of 420 ohms. Current was afforded by a storage battery of 110 volts. The rods were 30.5 cms. in length and 0.953 cm. in diameter. Tests were made with wrought iron, cast iron, steel, copper, brass, nickel, and aluminum.

Nature and Method of Measurement.

The maximum elongation for soft iron wire observed by Dr. More (See PHYSICAL REVIEW, Vol. III., No. 15) was 0.000175 cm. or about three wave-lengths of sodium light. By making the measurements therefore in wave-lengths we are provided with a direct method, and the measuring units are of about the same order of magnitude as the observed changes in length. When the mirror moves parallel to itself one-fourth of a wave-length, the cross wire of an observing telescope would appear to have passed over one uniformly colored space; if for example sodium light is used, one yellow space. The spaces may be easily read to tenths, or closer if one of the following methods is employed :

a. By using a micrometer microscope of proper focal distance and magnifying power.

b. By having a micrometer screw with a graduated head attached to the compensator.

c. By covering one half of the mirror with a scale which will divide the spaces into any desired number of parts.

By simple estimation the error of a single observation need not exceed 1 per cent., while a series of observations, employing one of the devices mentioned above would still farther reduce the error.

In case the change in the bar is sufficient to cause more than one fringe to pass the reference wire, fringes from white light should be employed. In this manner the identity of the fringes may be preserved.

Results.

The results obtained may be shown by the following tables which are followed by an explanation.

TABLE I.

SOFT IRON.

H.	*I.*	*B.*	*R.*	Maximum Elongation.	$\partial l/l$
5.5	183.0	2306	2920	0.020 λ	4×10^{-8}
17.6	689.6	8671	920	0.075 λ	15
31.1	977.8	12286	520	0.375 λ	75
34.4	1136.9	14296	470	0.425 λ	85
38.5	1343.3	16891	420	0.500 λ	100

TABLE II.

SOFT IRON.

No. of Successive Readings.	Direction.	Amount.
1	*e*	0.200 λ
1	*e*	0.175 λ
1	*e*	0.125 λ
1	*e*	0.125 λ
10	*e*	0.125 λ
2	*c*	0.200 λ
6	*e*	0.125 λ
2	*c*	0.250 λ
5	*e*	0.125 λ
10	*c*	0.250 λ
4	*e*	0.100 λ
1	*e*	0.075 λ
1	*c*	0.250 λ
3	*e*	0.200 λ
5	*e*	0.125 λ
1	*c*	0.250 λ

TABLE III.

JESSOP STEEL, RECTANGULAR SECTION.

Resistance.	Elongation.	Resistance.	Elongation.
420	0.90 λ	420	0.80 λ
470	0.60 λ	470	0.40 λ
520	0.50 λ	520	0.25 λ
620	0.45 λ	620	0.15 λ
920	0.25 λ	920	0.08 λ
1420	0.10 λ		
1920	0.05 λ		
420	0.40 λ	420	0.15 λ
470	0.25 λ	470	0.10 λ
520	0.15 λ	520	0.05 λ
620	0.05 λ		
420	0.10 λ		
470	0.05 λ		

In Table I. the attempt was made to find what relation existed between the magnetizing force and the magnetic intensity, and the elongation. As these values were in no case maximum values, the table merely shows that the latter increases with the former for the quantities used.

In making the observations for Table I. it was noticed after several successive readings had been taken that the behavior of the rod was somewhat abnormal. For example, after several elongations there would follow a *contraction*, and there seemed to be no method of predicting the nature or extent of the change. These occurred only after quite a number of readings which were regular. As these results were quite unexpected I tried the experiment on several occasions, and had the observations checked by an assistant. While no uniformity existed among the different readings, the abnormal action was always noted. In Table II., I have collected the results of a series of observations. Column 1 represents the number of observations which were uniform both in magnitude and direction; column 2 states the nature of the change—whether an elongation or a contraction; column 3 shows the magnitude of the change.

Table III. represents the results when currents decreasing in magnitude were successively sent around a rectangular bar of Jessop steel. Resistances varying from 420 to 1920 were placed in the circuit and the elongations noted; then these measurements were immediately repeated, so that in all five series were observed. The results obtained show:

a. Considerable regularity in the relation between current and elongation.

b. A regular decrease in elongation for the same current after repeated applications.

c. No abnormal effects as in the case of soft iron.

Distortion.

By use of interference methods, changes which might occur in a rod other than expansion or contraction may be noted. For example, if the rod is distorted the effect is very clearly seen. If the distortion is the result of horizontal planes slipping past each other

in a direction parallel to the length of the rod, the mirror will move as though rotated around a horizontal axis. In this case the fringes, which formerly appeared vertical, become inclined at an angle depending upon the amount of distortion. If the distortion is caused by a slipping of vertical planes, the mirror appears to rotate on a vertical axis, and the fringes approach or recede from each other. The first effect was quite noticeable in the case of soft iron, but did not appear in steel or nickel. The second was suspected in several observations but was more difficult to detect. No error is introduced in the method on account of the distortions as the extreme displacement of the fringes due to this cause would correspond with the maximum elongation or contraction sought.

Aluminum, copper, and brass gave no results that could be detected. Nickel showed a contraction for the values of H used of about $\lambda/4$. Upon making and breaking the circuit repeatedly it exhibited the abnormal action noticed in the case of soft iron. Cast iron expanded slightly and quite uniformly.

A test was made to determine whether the effect of change in temperature introduced an error. Bars of non-magnetic substances mentioned above showed no change in length due to heating effects, although a charge of $\lambda/40$ could readily have detected. Again the circuit was allowed to remain closed for some time until a change in temperature sufficient to be noticed resulted. The change was a *creeping* motion of the fringes which was in strong contrast with the sudden change observed when the bar was magnetized.

The initial contraction in metals which expand on magnetization, which has been reported by some observers, was not detected in this experiment. If it exists at all it must be characteristic of rods under certain conditions of stress, or containing certain peculiarities of structure not exhibited by the rods here used.

Having in the above investigations determined the best working conditions, and studied the behavior of various rods and the peculiarities of their state when magnetized, the experiment was assigned to a senior in the University, Miss Rena E. Dunn, to repeat the measurements under more favorable conditions. Dimensions were employed as follows:

Length of soft iron rod	75.5 cm.
Diameter	0.6 "
No. layers of wire in coil	13. "
No. turns to layer	1064. "
No. wire	25. "
Current through wire alone	0.5 amp.
Length of coil	84.8 cm.
External diameter	9 "
Internal diameter	4.7 "
No. of times per. cm.	163.1
Strength of field per amp.	205 C. G. S. units.
Maximum strength of field	102.5 " "

The coil being 9.3 cm. longer than the rod it may be assumed that the magnetizing force was used to good advantage in magnetizing the rod. As shown in the drawing the mirror was fastened on one end of the rod and placed within the coil, while the other end of the rod was joined to a strip of brass AC in the smaller figure, which in turn connected with the movable cylinder. For convenience a special frame was made for the remaining mirrors of the interferometer, but a description of this would not be necessary here. The current, as before, was attained from a 110-volt storage battery and was measured simultaneously with the optical measurements, by means of a Weston milli-voltmeter with a specially constructed shunt. Variation in resistance was produced by a bank of lamps.

In the following Table (IV.) ∂l represents the mean of twenty observations in the first case, and ten in each succeeding case with probable errors averaging 0.1%.

TABLE IV.

Current (Amps.).	H.	I.	B.	∂l	$\partial l/l$
0.500	102.5	1835.4	23173.1	13.37×10^{-5} cm.	17.7×10^{-7}
0.185	41.0	1153.2	14537.2	6.79	8.9
0.096	20.5	813.7	10249.1	4.66	6.2
0.061	12.8	620.6	7813.9	2.42	3.2
0.043	9.3	510.8	6430.3	1.55	2.1
0.028	5.7	446.9	5626.1	1.18	1.6
0.022	4.7	420.5	5290.3	1.11	1.5

Comparing these values for $\partial l/l$ with those obtained by Dr. More, it will be noticed that for small values of H the relative elongations

were much greater; while as H increased the elongations failed to increase in an equal ratio. His maximum was reached when $H = 37.6$, while in this case for $H = 102.5$ the elongation surpassed that for any smaller value.

In conclusion the following may be claimed as points of advantage passed by this method:

1. Since the mirror is practically a part of the rod, little or no annoyance due to vibrations of the building was experienced.

2. The measurements are direct and the unit so small that the error need not exceed one per cent.

3. The errors liable to occur in any mechanical device, such as the use of levers, indicators and the like, are eliminated.

4. Any auxiliary effects, such as distortion, are closely seen.

In addition to the work of Miss Dunn already mentioned, I ought to state that most of the mechanical arrangements for this investigation were made by Mr. L. H. Horner of the class of 1900, who also assisted me greatly in taking the observations.

PHYSICAL LABORATORY, UNIVERSITY OF MAINE.

AN HARMONIC ANALYZER.

By J. N. LeConte.

IN the study of the performance of the modern alternating current apparatus it is often important to have a more exact mathematical expression for electromotive force and current waves than that given by equations of the form,

$$y = a \sin (x + c)$$

which express such electromotive forces or currents as simple harmonic functions of the time. This is of especial importance in dealing with the waves produced by toothed armatures, and with the action of transformers and other induction apparatus where the efficiency may depend upon the form of the wave. Since alternator waves are always single-valued periodic curves, the method of expressing their equations as a series of sines and cosines of multiples of x according to Fourier's theorem naturally suggests itself as the simplest and most accurate for the purpose.

The harmonic analyzer herein described was originally designed to approximate to the first eleven terms of Fourier's series for any given alternator wave, but it may of course be used for analyzing any single-valued periodic curve, and the accuracy attained is certainly as great as that of most of the recent analyzers. The instrument can also be used for tracing, by a continuous process, the curve whose coefficients are given up to the eleventh term, though no great accuracy is claimed for this latter process.

The fundamental theory of the instrument is the same as that brought out by Mr. G. U. Yule, in 1895, the only difference being in the relative motion of the parts. The operation of Yule's Analyzer[1] is as follows; ABC (Fig. 1) is the curve to be analyzed, whose base length, AC, may be anything, but is fixed for any given instrument. It is best to make $AC = 2\pi$. DE is a rack mounted

[1] Philosophical Magazine, April, 1895.

on rollers like a parallel ruler whose edge is always parallel to AC, and can have no motion in the direction OC. P is the center of a gear which meshes with the rack and whose circumference is contained some exact number (n) times in AC. PS is an arm attached rigidly to the gear, and SRQ an ordinary planimeter, whose pole Q is fixed upon the drawing board. Then if the center P of the gear is originally at A, the starting point of the curve whose equation is $y = f(x)$, with the point s in the line OA and to the left of A, the area traced out by the point s as we move P around the curve ABC and back along the base line to A will be

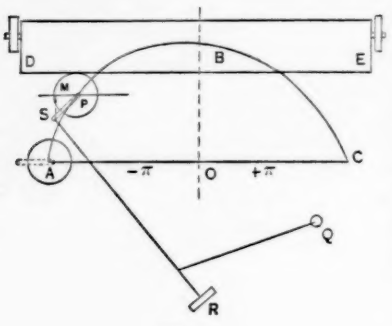

Fig. 1.

$$A = \int_{-\pi}^{+\pi} y dx + nr \cdot \cos \pi n \int_{-\pi}^{+\pi} y \cdot \sin nx \cdot dx$$

where $AC = 2\pi$, and r is the radius arm PS. The first integral is the area of the curve which we can cause to vanish by properly choosing our base line AC. In the second term $\cos \pi n = \pm 1$ according as n is even or odd. Under these conditions we have

$$A_1 = \pm \pi r n a_n$$

where a_n is the coefficient of a sine term of Fourier's series. A_1 is the reading of the planimeter, hence :—

$$a_n = \pm \frac{A_1}{\pi r n}$$

If in the initial position of the gear, the arm PS is perpendicular to OA and below A, then on completing the circuit about $ABCA$, the planimeter reading will be :—

$$A_2 = \pm \pi r n b_n$$

or

$$b_n = \pm \frac{A_2}{\pi r n}$$

where b_n is a coefficient of a cosine term of Fourier's series.

The principal limitation of the above form of apparatus, aside from the difficulty of manipulation, is the small number of terms which it will determine in any practical form of construction. It is evident that the least number of teeth on the gear which determines the first term or fundamental, must be that which contains exactly all the digits from unity up to the number of the highest term which is to be found. If we are to determine 7 terms we must have at least 420, and if eleven at least 27,720 teeth in the fundamental. Furthermore the gears for the higher terms become so small as to make the apparatus difficult to handle.

The range and ease of manipulation of the analyzer can be greatly extended by the following modifications: If we consider the motion of the gear P to be a combination of a rotation about P and a pure translation, we can allow the rotation to continue, but bring the translation to rest by giving to the curve, the drawing board, and planimeter pole an equal and opposite translation, and the action of the apparatus will in no way be changed. The center P being now a fixed point, we may introduce any chain of compound gearing between the gear and the rack and increase the number of coefficients which can be found to almost any extent. Finally if the tracing point of the planimeter, instead of being placed on S, is constrained to follow the motion of the point M, where M is the projection of S on a line parallel to AC, the result as given by the planimeter will not be affected.

An harmonic analyzer has recently been completed at the University of California according to these principles. Its construction and mode of operation are in outline as follows: A series of eleven axles (1, 2, 3, 4, Fig. 2) are connected by a chain of gearing in such a way that while the first revolves once, the second will revolve twice, the third three times and so on. Upon each axle is mounted a graduated circle which may be turned through any epoch angle and clamped in any position. Across a diameter of each circle is a sliding bar, carrying a cylindrical pin s, whose distance from the center P is adjusted by means of a graduated scale on the guides. Each pin works in the groove of a slotted cross-head to which it will communicate a simple harmonic motion when the whole is driven by the worm-wheel V. DK is a flat platform of

brass sliding along a pair of rectilinear guides DE and FG which are parallel to the direction of the harmonic motion of the cross-heads. Motion is communicated to it through a rack along DE by a system of gears from axle 1, so that while axle 1 rotates once, the platform will move through a distance 2π inches. Upon the upper face of this platform is another pair of rectilinear guides LK

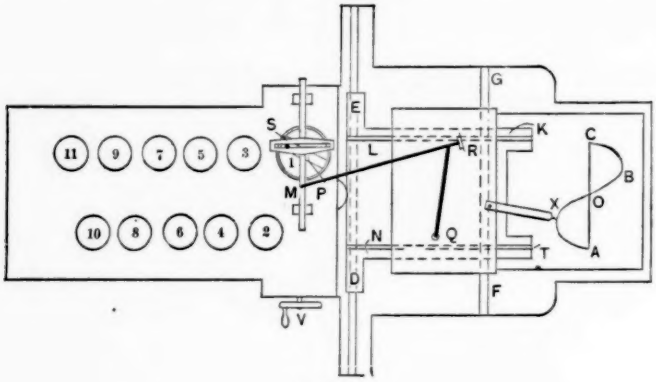

Fig. 2.

and NT and upon these a light carriage runs on grooved wheels carrying a small planimeter board. The curve to be analyzed is placed upon the base of the machine, and a point X attached to the carriage can be caused to follow around its perimeter.

In order to find the first sine coefficient of a curve ABC whose base length is 2π inches, the graduated circles are all turned until the sliding bars are parallel to one another and to the rack DE. A planimeter is now placed with its pole and roller upon the carriage and with its tracing point upon the cross-head at M. As the machine is driven by the handle at V, the point X starting from A is kept over the curve. Every point in the planimeter board will then describe the curve ABC, while the point M is given a simple harmonic motion parallel to AC. When X has arrived at C, the cross-head will have moved through one complete period. The rack is now thrown out of gear, and the whole platform slid back along its guides till X has returned to its starting point A. This removes the necessity of running the machine backwards which would introduce error due to back lash. The final reading of the roller R will

be proportional to a_1. By the same process a_2 is found by placing the planimeter point on the cross-head of 2 and so on. If the analysis is started with sliding bars of the dials at right angles to DE, the readings of the roller will be proportional to b_1, b_2, etc.

The accuracy of the instrument can be tested by analyzing a curve for which the integrals :

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} y \sin nx dx$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} y \cos nx dx$$

can be evaluated. In the case of the zigzag straight line (Fig. 3) where $y = x$

$$a_n = -\frac{1}{\pi}\frac{(-1)^n 2\pi}{n}$$

and $b_n = 0$.

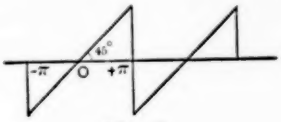

Fig. 3.

The following are the results given by the analyzer for 11 consecutive runs, where a small polar planimeter was used reading only to hundredths of a square inch.

No.	Calculated.	Observed.	Error.	Per cent. of a_n.	Per cent. of a_1.
a_1	+ 2.	+ 1.983	.017	0.85%	0.85
a_2	− 1.	− 0.996	.004	0.40	0.20
a_3	+ 0.666	+ 0.667	.001	0.15	0.05
a_4	− 0.5	− 0.497	.003	0.60	0.15
a_5	+ 0.4	+ 0.397	.003	0.75	0.15
a_6	− 0.333	− 0.332	.001	0.30	0.05
a_7	+ 0.286	+ 0.289	.003	1.05	0.15
a_8	− 0.25	− 0.251	.001	0.40	0.05
a_9	+ 0.222	+ 0.223	.001	0.44	0.05
a_{10}	− 0.2	− 0.201	.001	0.50	0.05
a_{11}	+ 0.182	+ 0.183	.001	0.66	0.05

No difficulty whatever is experienced in driving the handle of the worm-wheel with one hand and guiding the point X around the curve with the other. In the case of an irregular alternator wave whose double amplitude was six inches, ten consecutive runs around the same curve under the same conditions gave the following planimeter readings, the point X being guided wholly by hand.

No.	Planimeter reading.
1	2.99
2	3.02
3	3.00
4	3.00
5	3.00
6	3.00
7	3.00
8	3.01
9	3.02
10	3.00
	3.004 Mean.

In this case the probable error of a measurement was .0065 or 0.22%, while that of the arithmetic mean is .0020 or only 0.07%.

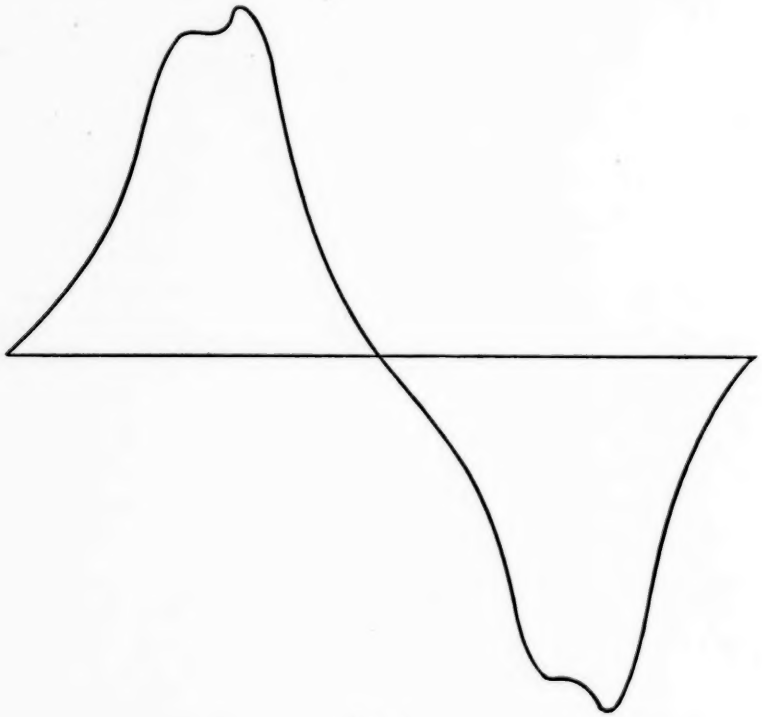

Fig. 4.

The series of eleven cross-heads moving parallel with one another with motions simply harmonic and representing all periods

from 1 to 11, suggests a method of using the machine as a curve tracer after the manner of Lord Kelvin's tide recording apparatus. Accordingly a grooved pulley is mounted upon each cross-head, and a No. 40 brass wire is passed around them all. The free end is attached to a light pen carrier which slides on a pair of tense wires at right angles to *DE*. A small weight takes up the slack of the fine wire, and the drawing paper is fastened to the platform *DK*. Amplitudes are set off on the sliding bars of the elements, and epoch angles on the graduated circles. The great advantage of this arrangement is that the curve represented by the equation

$$y = \tfrac{1}{2} c_0 \sin \beta_0 + c_1 \sin (x + \beta_1) + c_2 \sin (2x + \beta_2) + c_3 \sin (3x + \beta_3) + \cdots$$

can be drawn directly, and not merely the components represented by the sine and cosine series. Fig. 4 shows an electromotive force wave taken by the method of instantaneous contact from a small single

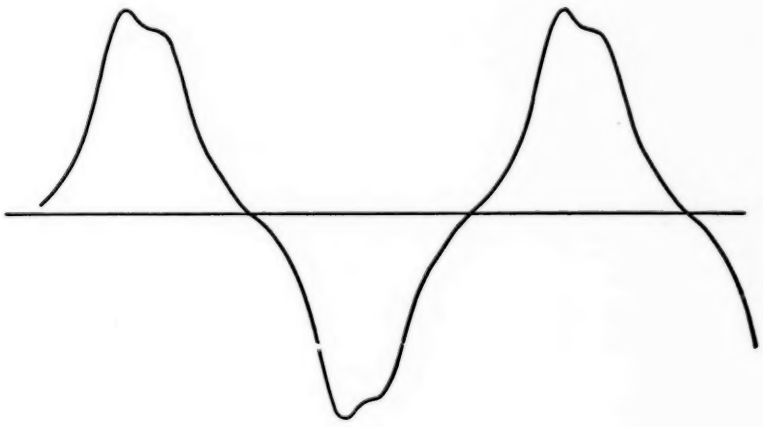

Fig. 5.

phase alternator with toothed armature, and also the analysis of the same as given by the analyzer. Fig. 5 shows the curve as reproduced on the machine by a single operation. The slight difference is due to the small number of terms represented.

No.	Sine Components.	Cosine Components.
1	+2.435	−0.295
3	−0.452	+0.263
5	−0.065	−0.040
7	+0.095	−0.071
9	−0.038	+0.064
11	+0.003	−0.007

No.	Amplitude.	Epoch Angle.
1	2.453	− 6° 55′
3	.523	+149° 48′
5	.076	−148° 24′
7	.118	−360° 46′
9	.074	+121° 6′
11	.008	− 66° 48′

Le Conte : An Harmonic Analyzer.
(Arranged as Curve-tracer.)

U. OF MICH.

A QUANTITATIVE STUDY OF THE HIGH-FREQUENCY INDUCTION COIL.

By W. P. Boynton.

THE behavior of circuits containing self- and mutual-induction, and some of the properties of the disruptive discharge of a Leyden jar were experimentally investigated by Faraday[1] and Joseph Henry.[2] The latter, besides studying the action of two mutually inducing coils, one of which was traversed by the current from a galvanic battery, also discharged a Leyden jar through one coil, and noted the currents induced in the neighboring coil.[3] Henry also noted phenomena in the discharge of the Leyden jar from which he concluded that this must be oscillatory in character.[4]

Perhaps the first direct experimental verification of the oscillatory character of this discharge was made by Feddersen in the years 1857–1862,[5] by observing the spark in a revolving mirror.

The mathematical theory of the oscillatory discharge was given by Lord Kelvin[6] and Kirchhoff.[7] The general theory of electrical oscillations has been discussed at more or less length, particularly by J. J. Thomson.[8]

Hittorf,[9] J. J. Thomson[10] and others have used the discharge of

[1] Experimental Researches, Series I., Induction of Electric Currents. Series XII., The Disruptive Discharge.

[2] Smithsonian Miscellaneous Collections: Scientific Writings of Joseph Henry; Contributions to Electricity and Magnetism; III. (1838) p. 108, and IV. (1840), On Electromagnetic Induction; V. (1842) p. 200, On Electrostatic Induction and the Oscillatory Discharge.

[3] Ibid., pp., 132 et seq.

[4] Ibid., p., 200.

[5] Pogg. Ann. 103, p. 69, 1858; 108, p. 497, 1859; 112, p. 452, 1861; 113, p. 437, 1861; 116, p. 132, 1862.

[6] Phil. Mag. (4), 5, p. 393, 1853. Math. and Phys. Papers, Vol. I., p. 540.

[7] Pogg. Ann. 121, p. 551, 1864. Ges. Abh., p. 168.

[8] Recent Researches in Electricity and Magnetism, Chap. IV., p. 251.

[9] Wied. Ann. 21, p. 90, 1864.

[10] Proc. Roy. Soc., 45, p. 269, 1869. Recent Researches, p. 92.

a Leyden jar through a coil of wire about a vacuum tube as a means of studying the behavior of the rarefied gas. Within the present decade many have repeated Henry's experiment of discharging a condenser through one coil of wire and studying the effect produced in a secondary circuit. Many[1] have paid particular attention to the phenomena of the varying electrostatic field, while some have attempted to follow the behavior of the current in one of the circuits. Colley[2] has accomplished this by making the period very slow, and using an "oscillometer," either a fluctuating flame, or the mirror of an exceedingly sensitive galvanometer.

THEORETICAL.

The mathematical theory of the so-called Tesla coil has been discussed by Oberbeck and others.[3] The discussion which follows is a modification and extension of Oberbeck's. It is essentially a discussion of a system with two degrees of freedom.

Let the suffixes 1 and 2 refer to the primary and secondary circuits respectively. Let V represent the difference of potential at the terminals of the condenser, I the current, Q the charge on the condenser, L self-induction, M mutual-induction, K capacity, R resistance. We have then to distinguish two cases: (1) The secondary circuit is open, that is, closed by a capacity. (2) The secondary circuit is closed by a resistance, or short circuited.

1. The differential equations of the system are:

$$V_1 + L_1 \frac{dI_1}{dt} + M\frac{dI_2}{dt} + R_1 I_1 = 0,$$

$$V_2 + L_2 \frac{dI_2}{dt} + M\frac{dI_1}{dt} + R_2 I_2 = 0.$$

[1] Nikola Tesla, Electrical Engineer (N. Y.), XII., p. 35, 1891; XV., pp. 42, 65, 88, 531, 553, 579, 603, 626, 1893. Published also in book form. Elihu Thomson, Elec. Eng., XIII., pp. 159, 199, 1892. H. Ebert, Wied. Ann. 53, p. 144, 1894. Elster, 10, Jahresber. des Ver. f. Naturwiss. zu Braunschweig, p. 43, 1895; Wied. Beibl. 20 p. 338, 1896.

[2] Wied. Ann. 26, p. 432, 1895; 28, p. 1, 1886; 44, p. 102, 1891. See also Hotchkiss and Millis, Phys. Rev. 3, p. 49, 1895.

[3] Oberbeck, Wied. Ann. 55, p. 623, 1895. Domalip and Kolácek, Wied. Ann. 57, p. 731, 1896. Blümcke, Wied. Ann. 58, p. 405, 1896. Wien. Wied. Ann. 61, p. 151, 1897, gives the mathematical theory for two circuits in the general case, where they have not only mutual induction, but also mutual capacity and "mutual resistance," that is, affect the distribution of the current in the bodies of the conductors.

Making use of the equations of continuity,

$$I = \frac{dQ}{dt} \text{ and of charge, } V = \frac{Q}{K}, \text{ these become}$$

$$\frac{Q_1}{K_1} + L_1\frac{d^2Q_1}{dt^2} + M\frac{d^2Q_2}{dt^2} + R_1\frac{dQ_1}{dt} = 0,$$

$$\frac{Q_2}{K_2} + L_2\frac{d^2Q_2}{dt^2} + M\frac{d^2Q_1}{dt^2} + R_2\frac{dQ_2}{dt} = 0.$$

Assuming that the solutions are of the form

$$Q_1 = e^{\lambda t},$$

$$Q_2 = ke^{\lambda t},$$

our equations become

$$1 + L_1K_1\lambda^2 + kMK_1\lambda^2 + R_1K_1\lambda = 0$$

$$k + kL_2K_2\lambda^2 + MK_2\lambda^2 + kR_2K_2\lambda = 0.$$

Solving for k,

$$(1) \qquad k = -\frac{1 + L_1K_1\lambda^2 + R_1K_1\lambda}{MK_1\lambda^2} = -\frac{MK_2\lambda^2}{1 + L_2K_2\lambda^2 + R_2K_2\lambda}.$$

Clearing of fractions, collecting and dividing by coefficient of λ^4,

$$\lambda^4 + \lambda^3\frac{L_1R_2 + L_2R_1}{L_1L_2 - M^2} + \lambda^2\frac{L_1K_1 + L_2K_2 + R_1R_2K_1K_2}{K_1K_2(L_1L_2 - M^2)}$$

$$+ \lambda\frac{R_1K_1 + R_2K_2}{K_1K_2(L_1L_2 - M^2)} + \frac{1}{K_1K_2(L_1L_2 - M^2)} = 0,$$

which is of the fourth degree in λ. We are interested in the imaginary roots, which occur, if at all, in conjugate pairs, since the coefficients of our equation are all real. If we write these

$$\lambda_1 = -\alpha + i\beta, \qquad \lambda_3 = -\gamma + i\delta,$$

$$\lambda_2 = -\alpha - i\beta, \qquad \lambda_4 = -\gamma - i\delta,$$

then by the theory of equations

$$(2) \qquad -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = 2\alpha + 2\gamma = \frac{L_2R_1 + L_1R_2}{L_1L_2 - M^2} = A,$$

$$(3)\quad \sum_{(r+s)}\lambda_r\lambda_s = \lambda_1\lambda_2 + \lambda_3\lambda_4 + (\lambda_1+\lambda_2)(\lambda_3+\lambda_4) = \alpha^2+\beta^2+\gamma^2+\delta^2+4\alpha\gamma$$
$$= \frac{L_1K_1 + L_2K_2 + R_1R_2K_1K_2}{K_1K_2(L_1L_2 - M^2)} = B,$$

$$(4)\quad -\sum_{(r+s+t)}\lambda_r\lambda_s\lambda_t = -(\lambda_1+\lambda_2)\lambda_3\lambda_4 - (\lambda_3+\lambda_4)\lambda_1\lambda_2 = 2\alpha(\gamma^2+\delta^2) + 2\gamma(\alpha^2+\beta^2$$
$$= \frac{R_1K_1 + R_2K_2}{K_1K_2(L_1L_2 - M^2)} = C,$$

$$(5)\quad \lambda_1\lambda_2\lambda_3\lambda_4 = (\alpha^2+\beta^2)(\gamma^2+\delta^2) = \frac{1}{K_1K_2(L_1L_2 - M^2)} = D.$$

From (3) and (5), disregarding $4\alpha\gamma$ as small, we get

$$(6)\quad \alpha^2+\beta^2 = \frac{B+\sqrt{B^2-4D}}{2};\quad \gamma^2+\delta^2 = \frac{B-\sqrt{B^2-4D}}{2}$$

and from these and (2) and (4)

$$(7)\quad \alpha = \frac{A}{4} + \frac{AB-2C}{4\sqrt{B^2-4D}};\quad \gamma = \frac{A}{4} - \frac{AB-2C}{4\sqrt{B^2-4D}}.$$

By making the proper substitutions these become

$$(8)\quad \alpha^2+\beta^2 = \frac{L_1K_1 + L_2K_2 + \sqrt{(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2}}{2K_1K_2(L_1L_2 - M^2)}$$

$$(9)\quad \gamma^2+\delta^2 = \frac{L_1K_1 + L_2K_2 - \sqrt{(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2}}{2K_1K_2(L_1L_2 - M^2)}$$

$$(10)\quad \alpha = \frac{R_1L_2 + R_2L_1 + \dfrac{R_1[L_2(L_1K_1+L_2K_2) - 2K_1(L_1L_2-M^2)] + R_2[L_1(L_1K_1+L_2K_2) - 2K_2(L_1L_2-M^2)]}{\sqrt{(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2}}}{4(L_1L_2 - M^2)}$$

$$(11)\quad \gamma = \frac{R_1L_2 + R_2L_1 - \dfrac{R_1[L_2(L_1K_1+L_2K_2) - 2K_1(L_1L_2-M^2)] + R_2[L_1(L_1K_1+L_2K_2) - 2K_2(L_1L_2-M^2)]}{\sqrt{(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2}}}{4(L_1L_2 - M^2)}$$

The general solution may now be written in exponential form

$$Q_1 = Ee^{\lambda_1 t} + Fe^{\lambda_2 t} + Ge^{\lambda_3 t} + He^{\lambda_4 t},$$
$$Q_2 = k_1 Ee^{\lambda_1 t} + k_2 Fe^{\lambda_2 t} + k_3 Ge^{\lambda_3 t} + k_4 He^{\lambda_4 t},$$

which reduces to the trigonometrical form

$$Q_1 = e^{-\alpha t}[(E+F)\cos\beta t + i(E-F)\sin\beta t] + e^{-\gamma t}[(G+H)\cos\delta t + i(G-H)\sin\delta t],$$
$$Q_2 = e^{-\alpha t}[(k_1E+k_2F)\cos\beta t + i(k_1E-k_2F)\sin\beta t] + e^{-\gamma t}[(k_3G+k_4H)\cos\delta t + i(k_3G-k_4H)\sin\delta t],$$

or otherwise

$$Q_1 = e^{-\alpha t}(A_1\cos\beta t + B_1\sin\beta t) + e^{-\gamma t}(C_1\cos\delta t + D_1\sin\delta t),$$
$$Q_2 = e^{-\alpha t}(A_2\cos\beta t + B_2\sin\beta t) + e^{-\gamma t}(C_2\cos\delta t + D_2\sin\delta t).$$

Equating the coefficients of corresponding terms, eliminating E, F, G, H, and noting that we may write

$$k_1 = a + bi, \quad k_3 = c + di, \qquad (12)$$
$$k_2 = a - bi, \quad k_4 = c - di,$$

we obtain the four relations

$$aA_1 + bB_1 = A_2, \quad -bA_1 + aB_1 = B_2, \qquad (13)$$
$$cC_1 + dD_1 = C_2, \quad -dC_1 + cD_1 = D_2.$$

The initial conditions that, when $t = 0$,

$$Q_1 = V_1K_1 = V_0K_1, \quad \frac{dQ_1}{dt} = 0, \quad Q_2 = 0, \quad \frac{dQ_2}{dt} = 0,$$

give the four equations

$$A_1 + C_1 = V_0K_1, \quad -\alpha A_1 + \beta B_1 - \gamma C_1 + \delta D_1 = 0, \qquad (14)$$
$$A_2 + C_2 = 0, \quad -\alpha A_2 + \beta B_2 - \gamma C_2 + \delta D_2 = 0,$$

which suffice with equations (13) to determine all the eight constants.

If from equations (14) we eliminate A_2, B_2, C_2, D_2, by equations (13) we have four equations in A_1, B_1, C_1, D_1.

$$
\begin{aligned}
A_1 \quad + \quad C_1 \quad &= V_0 K_1, \\
aA_1 + bB_1 + cC_1 + dD_1 &= 0, \\
-aA_1 + \beta B_1 - \gamma C_1 + \delta D_1 &= 0, \\
(a\alpha + \beta b)A_1 + (ab - \beta a)B_1 + (\gamma c + \delta d)C_1 + (\gamma d - \delta c)D_1 &= 0,
\end{aligned} \tag{15}
$$

whose determinant is

$$\Delta = \beta\delta(a^2 + b^2 + c^2 + d^2) - bd(a^2 + \beta^2 + \gamma^2 + \delta^2) + 2a\gamma bd - 2\beta\delta ac$$

and their solution

$$
\begin{aligned}
A_1 &= \frac{V_0 K_1[\beta\delta(c^2 + d^2) - bd(\gamma^2 + \delta^2) + (ab - \beta a)(\gamma d + \delta c)]}{\Delta} \\
B_1 &= \frac{V_0 K_1[a\delta(c^2 + d^2) + ad(\gamma^2 + \delta^2) - (a\alpha + \beta b)(\gamma d + \delta c)]}{\Delta} \\
C_1 &= \frac{V_0 K_1[\beta\delta(a^2 + b^2) - bd(a^2 + \beta^2) + (ab + \beta a)(\gamma d - \delta c)]}{\Delta} \\
D_1 &= \frac{V_0 K_1[\beta\gamma(a^2 + b^2) + bc(a^2 + \beta^2) - (ab + \beta a)(\gamma c + \delta d)]}{\Delta}
\end{aligned} \tag{16}
$$

or, if we disregard the squares of the small quantities a, γ, b, d, (b/a is of the order of a/β, etc.),

$$
\begin{aligned}
\Delta &= \beta\delta(a - c)^2 \\
A_1 &= \frac{V_0 K_1 \beta\delta(c^2 - ac)}{\Delta} = \frac{V_0 K_1 c}{c - a}; \\
B_1 &= \frac{V_0 K_1[a\delta c^2 + ad\delta^2 - \delta c(a\alpha + \beta b)]}{\Delta} \\
C_1 &= \frac{V_0 K_1 \beta\delta(a^2 - ac)}{\Delta} = \frac{V_0 K_1 a}{a - c}; \\
D_1 &= \frac{V_0 K_1[\beta\gamma a^2 + bc\beta^2 - \beta a(\gamma c + \delta d)]}{\Delta}
\end{aligned} \tag{17}
$$

where B_1, and D_1, are small of the first order in comparison with A_1, and C_1,

a and b of equation (12) are the real part and the coefficient of the imaginary part of k_1, respectively. From (1) we have

$$k_1 = -\frac{1+L_1K_1\lambda_1^2+R_1K_1\lambda_1}{MK_1\lambda_1^2}$$

$$= -\frac{1+L_1K_1(-\alpha+i\beta)^2+R_1K_1(-\alpha+i\beta)}{MK_1(-\alpha+i\beta)^2}$$

$$= -\frac{(-\alpha-i\beta)^2+L_1K_1(\alpha^2+\beta^2)^2+R_1K_1(-\alpha-i\beta)(\alpha^2+\beta^2)}{MK_1(\alpha^2+\beta^2)^2}$$

from which we get

$$a = -\frac{(\alpha^2-\beta^2)+L_1K_1(\alpha^2+\beta^2)^2-R_1K_1\alpha(\alpha^2+\beta^2)}{MK_1(\alpha^2+\beta^2)^2}$$

$$b = -\frac{2\alpha\beta-R_1K_1\beta(\alpha^2+\beta^2)}{MK_1(\alpha^2+\beta^2)^2}$$

If α is so small that we may disregard its square in comparison with β, these become

$$a = \frac{1-L_1K_1\beta^2}{MK_1\beta^2} = \frac{\delta^2-L_1K_1\beta^2\delta^2}{MK_1\beta^2\delta^2},$$

(18)

$$b = \frac{R_1K_1\beta^2-2\beta}{MK_1\beta^3};$$

and by substituting γ and δ for α and β,

$$c = \frac{1-L_1K_1\delta^2}{MK_1\delta^2} = \frac{\beta^2-L_1K_1\beta^2\delta^2}{MK_1\beta^2\delta^2}$$

(18)

$$d = \frac{R_1K_1\delta^2-2\gamma}{MK_1\delta^3}$$

Substituting the values of β and δ we get a and c

$$a = \frac{L_2K_2-L_1K_1-\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}}{2MK_1}$$

$$c = \frac{L_2K_2-L_1K_1+\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}}{2MK_1}$$

Substituting these values in Equation (17) we have

(19) $$A_1 = \frac{1-X}{2}V_0K_1; \qquad C_1 = \frac{1+X}{2}V_0K_1$$

where $$X = \frac{L_1K_1-L_2K_2}{\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}}$$

and (20) $$-A_2 = C_2 = \frac{ac}{a-c} V_0 K_1 = \frac{MK_1K_2V_0}{\sqrt{(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2}}$$

B_2 and D_2 are small of the first order in comparison with A_2 and C_2.

II. If the secondary circuit be closed, V_2 drops out, or K_2 may be considered infinite, and our equation (1) becomes

$$k = -\frac{1 + L_1K_1\lambda^2 + R_1K_1\lambda}{MK_1\lambda^2} = -\frac{M\lambda^2}{L_2\lambda^2 + R_2\lambda} = -\frac{M\lambda}{L_2\lambda + R_2}$$

whence $$\lambda_1 = -\alpha + i\beta, \quad \lambda_2 = -\alpha - i\beta, \quad \lambda_3 = -\gamma$$

where $$\alpha = \tfrac{1}{2}\left[\frac{L_1R_2 + L_2R_1}{L_1L_2 - M^2} - \frac{R_2}{L_2}\right] = \frac{L_2^2R_1 + M^2R_2}{2L_2(L_1L_2 - M^2)}$$

(21)
$$\beta = \sqrt{\frac{L_2}{K(L_1L_2 - M^2)}}$$

$$\gamma = \frac{R_2}{L_2}$$

$$a = -\frac{M}{L_2}$$

$$b = \beta\left[\frac{R_1K(L_1L_2 - M^2)}{ML_2} - \frac{2\alpha K(L_1L_2 - M^2)^2}{ML_2^2}\right]$$

$$= -\beta\frac{R_2MK(L_1L_2 - M^2)}{L_2^3}$$

$$c = -\frac{L_2^2}{R_2^2MK}$$

The general solution may be written

$$Q_1 = e^{-\alpha t}(A_1\cos\beta t + B_1\sin\beta t) + C_1e^{-\gamma t},$$
$$Q_2 = e^{-\alpha t}(A_2\cos\beta t + B_2\sin\beta t) + C_2e^{-\gamma t}.$$

Where the constants are related by the equations

$$aA_1 + bB_1 = A_2,$$
$$aB_1 + bA_1 = B_2,$$
$$cC_1 = C_2.$$

And subject to the initial conditions that, when $t = 0$,

$$V_1 = V_0, \quad \frac{dV_1}{dt} = 0, \quad I_2 = 0,$$

or

$$\begin{aligned} A_1 \qquad + C_1 &= V_0 K, \\ -\alpha A_1 + \beta B_1 - \gamma C_1 &= 0, \\ -\alpha A_2 + \beta B_2 - \gamma C_2 &= 0, \end{aligned}$$

which completely determine the constants. Substituting the values of A_2, B_2, C_2, in the last equation we get

$$\begin{aligned} A_1 \qquad\qquad + C_1 &= V_0 K, \\ -\alpha A_1 \qquad + \beta B_1 \quad - \gamma C_1 &= 0, \\ (\alpha a + \beta b) A_1 + (\alpha b - \beta a) B_1 + \gamma c C_1 &= 0, \end{aligned} \tag{22}$$

whose determinant is

$$\begin{aligned} \Delta &= -b(\alpha^2 + \beta^2 - a\gamma) + \beta\gamma(c - a), \\ A_1 &= \frac{V_0 K}{\Delta} \gamma(\beta c + \alpha b - \beta a), \\ B_1 &= \frac{V_0 K}{\Delta} \gamma(\alpha c - \alpha a - \beta b), \\ C_1 &= \frac{V_0 K}{\Delta} b(-\alpha^2 - \beta^2), \\ A_2 &= \frac{V_0 K}{\Delta} \gamma[c(\alpha b + \beta a) - \beta(a^2 + b^2)], \\ B_2 &= \frac{V_0 K}{\Delta} \gamma[c(\alpha a - \beta b) - \alpha(a^2 + b^2)], \\ C_2 &= \frac{V_0 K}{\Delta} bc(-\alpha^2 - \beta^2). \end{aligned} \tag{23}$$

The quantities which are observable and measureable in the ordinary type of instruments are not the instantaneous potentials and currents, whose values we have just deduced, but the "effective" values, that is, the square roots of the mean squares. It is desirable then to evaluate an integral of the form $\int V^2 dt$ for the case of a single-damped oscillation, and also for two superposed oscillations. By giving the proper values to certain constants this will include all the cases which we shall need to consider.

a. The general exponential form for an oscillation of any amplitude and period is

$$V = Ee^{\lambda t} + Fe^{\mu t}$$

where E and F may be complex, and

$$\lambda = -\alpha + \beta i,\ \mu = -\alpha - \beta i,$$

where α and β are real and greater than 0. Then

$$\int V^2 dt = E^2 \int e^{2\lambda t}\, dt + F^2 \int e^{2\mu t} + 2EF \int e^{(\lambda+\mu)t}\, dt,$$

$$= \frac{E^2 e^{2\lambda t}}{2\lambda} + \frac{F^2 e^{2\mu t}}{2\mu} + \frac{2EFe^{(\lambda+\mu)t}}{\lambda+\mu}$$

$$= \frac{E^2 \mu e^{2\lambda t} + F^2 \lambda e^{2\mu t}}{2\lambda\mu} + \frac{2EFe^{(\lambda+\mu)t}}{\lambda+\mu}$$

where all the denominators are real, or in terms of α and β,

$$\int V^2 dt = \frac{e^{-2\alpha t}[E^2(-\alpha+\beta i) + F^2(-\alpha-\beta i))\cos 2\beta t + i(E^2(-\alpha+\beta i) - F^2(-\alpha-\beta i))\sin 2\beta t]}{2(\alpha^2+\beta^2)} + \frac{2EFe^{-2\alpha t}}{-2\alpha}$$

Since the oscillation is real

$$V = e^{-\alpha t}[(E+F)\cos\beta t + i(E-F)\sin\beta t]$$
$$\equiv e^{-\alpha t}(A\cos\beta t + B\sin\beta t.)$$

substituting A and B from this identity the imaginary parts vanish.

$$\int V^2\, dt = \frac{e^{-2\alpha t}[((B^2-A^2)\alpha + 2AB\beta)\cos 2\beta t + ((B^2-A^2)\beta - 2AB\alpha)\sin 2\beta t]}{4(\alpha^2+\beta^2)} - \frac{(A^2+B^2)e^{-2\alpha t}}{4\alpha}.$$

If, now, β is large in comparison with α, the first term may be disregarded in comparison with the last, and in particular

$$\int_0^\infty V^2\, dt = \frac{A^2+B^2}{4\alpha}. \tag{24}$$

b. Of two superimposed oscillations each gives in the integral terms of the form deduced above, but the terms arising from the cross products of terms with different periods and decrements require especial investigation. Such a typical term is

$$\int MNe^{(\mu+\nu)t}\,dt = \frac{MNe^{(\mu+\nu)t}}{\mu+\nu},$$

Where

$$M = \frac{A+Bi}{2}; \quad N = \frac{C+Di}{2}; \quad \mu = -\alpha+\beta i; \quad \nu = -\gamma+\delta i.$$

The sum of all such integrated terms, if V be of the form

$$V = e^{-\alpha t}(A\cos\beta t + B\sin\beta t) + e^{-\gamma t}(C\cos\delta t + D\sin\delta t)$$

reduces to the trigonometrical form is,

$$e^{-(\alpha+\gamma)t}\left\{\frac{-[(AC-BD)(\alpha+\gamma)+(BC+AD)(\beta+\delta)]\cos(\beta+\delta)t + [-(AC-BD((\beta+\delta)-(BC+AD)(\alpha+\gamma)]\sin(\beta+\delta)t}{(\alpha+\gamma)+(\beta+\delta)}\right.$$

$$\left. + \frac{-[(AC+BD)(\alpha+\gamma)+(BC-AD)(\beta-\delta)]\cos(\beta-\delta)t + [(AC+BD)(\beta-\delta)-(BC-AD)(\alpha+\gamma)]\sin(\beta-\delta)t}{(\alpha+\gamma)^2+(\beta-\delta)^2}\right\},$$

Which, taken between the limits 0 and ∞ is

$$\frac{(AC-BD)(\alpha+\gamma)+(BC+AD)(\beta+\delta)}{(\alpha+\gamma)^2+(\beta+\delta)^2} + \frac{(AC+BD)(\alpha+\gamma)+(BC-AD)(\beta-\delta)}{(\alpha+\gamma)^2+(\beta-\delta)^2},$$

or, if α and γ are so small that they can be neglected in comparison with β and δ,

$$\frac{BC+AD}{\beta+\delta} + \frac{BC-AD}{\beta-\delta}$$

which is ordinarily small in comparison with the principal terms, and can be neglected.

c. V is the sum of harmonic and oscillatory terms. The preceding discussion of case *b* is immediately applicable, by putting $\gamma = 0$. In general, also, the period of the oscillation is so much less

than that of the harmonic terms that δ is negligible in comparison with β, and our last expression reduces to $\frac{2BC}{\beta}$, which is entirely negligible in comparison with the principal terms.

In the case of the potential in the secondary circuit of our apparatus

$$\int_0^\infty V_2^2 dt = \frac{A_2^2 + B_2^2}{4\alpha K_2^2} + \frac{C_2^2 + D_2^2}{4\gamma K_2^2},$$

which becomes, neglecting B_2 and D_2, and noting that $A_2^2 = C_2^2$,

$$\frac{A_2^2}{4K_2^2}\left(\frac{1}{\alpha} + \frac{1}{\gamma}\right) = \frac{A_2^2(\alpha + \gamma)}{4\alpha\gamma K_2^2}.$$

Substituting the values of A_2 from equation (20), and of α and γ from equations (10) and (11), or directly from equations (2)–(7), rationalizing and performing the necessary algebraic simplifications we get

$$\int_0^a V_2^2 dt = \frac{V_0^2 M^2 K_1^2 (R_1 L_2 + R_2 L_1)}{2[R_1 R_2 (L_1 K_1 - L_2 K_2)^2 + M^2 (R_1 K_1 + R_2 K_2)^2]}.$$

The "effective" potential squared will be this quantity multiplied by $2n$ where n is the frequency of the alternating current charging the condenser, or calling this $\overline{V_2^2}$

$$(25) \qquad \overline{V_2^2} = \frac{n V_0^2 M^2 K_1^2 (R_1 L_2 + R_2 L_1)}{R_1 R_2 (L_1 K_1 - L_2 K_2)^2 + M^2 (R_1 K_1 + R_2 K_2)^2}.$$

The general expression for the current in either circuit is

$$I = \frac{dQ}{dt} = \left[\begin{array}{l} e^{-\alpha t}((-\alpha A + \beta B)\cos\beta t + (-\alpha B - \beta A)\sin\beta t) \\ + e^{-\gamma t}((-\gamma C + \delta D)\cos\delta t + (-\gamma D - \delta C)\sin\delta t) \end{array}\right]$$

$$\int_0^\infty I^2 dt = \frac{(-\alpha A + \beta B)^2 + (\alpha B + \beta A)^2}{4\alpha} + \frac{(-\gamma C + \delta D)^2 + (\gamma D + \delta C)^2}{4\gamma}$$

$$= \frac{(\alpha^2 + \beta^2)(A^2 + B^2)}{4\alpha} + \frac{(\gamma^2 + \delta^2)(C^2 + D^2)}{4\gamma}$$

which becomes, neglecting α^2, γ^2, B^2, and D^2 as small

$$\int_0^\infty I^2 dt = \frac{\beta^2 A^2}{4\alpha} + \frac{\delta^2 C^2}{4\gamma}.$$

Applying this to circuit 2 where $A_2^2 = C_2^2$ and substituting and reducing as before, we get

$$\int_0^\infty I_2^2 dt = \frac{K_1 K_2 V_0^2 M^2 (R_1 K_1 + R_2 K_2}{2\,[R_1 R_2 (L_1 K_1 - L_2 K_2)^2 + M^2 (R_1 K_1 + R_2 K_2)^2]},$$

and the "effective" current squared is

$$(26) \qquad \bar{I}_2^2 = \frac{n V_0^2 M^2 K_1 K_2 (R_1 K_1 + R_2 K_2)}{R_1 R_2 (L_1 K_1 - L_2 K_2) + M^2 (R_1 K_1 + R_2 K_2)^2}.$$

In the case of the primary circuit we shall see that with our arrangement the coefficient C_1, decidedly preponderates over the others. Then we have

$$\int_0^2 I_1^2 dt = \frac{\partial^2 C_1^2}{4\gamma}$$

$$(27) \quad \begin{aligned} &= C_1^2[(R_1 K_1 + R_2 K_2)\,((L_1 K_1 - L_2 K_2)^2 + 4M^2 K_1 K_2) - \\ &(L_1 K_1 - L_2 K_2)(R_1 K_1 - R_2 K_2)\sqrt{(L_1 K_1 - L_2 K_2)^2 + 4M^2 K_1 K_2)}] \\ &+ 4K_1 K_2\,[R_1 R_2 (L_1 K_1 - L_2 K_2)^2 + M^2 (R_1 K_1 - R_2 K_2)^2] \end{aligned}$$

and $\bar{I}_1^2$, is this expression multiplied as usual by $2n$.

An interesting approximation is obtained when $R_2 K_2$ is small in comparison with R_1, K_1, and is disregarded. Our three formulæ just obtained then become

$$(25') \qquad \bar{V}_2^2 = \frac{n V_0^2 M^2 K_1^2 \left(L_2 + L_1 \frac{R_2}{R_1}\right)}{R_2 (L_1 K_1 - L_2 K_2)^2 R_1 + M_1^2 K^2}$$

$$(26') \qquad \bar{I}_2^2 = \frac{n V_0^2 M^2 K_1^2 K_2}{R_2 (L_1 K_1 - L_2 K_2)^2 + R_1 M^2 K_1^2}$$

$$(27) \quad \bar{I}_1^2 = \frac{\begin{aligned} &nC_1^2[(L_1 K_1 - L_2 K_2)^2 + 4M^2 K_1 K_2 \\ &- (L_1 K_1 - L_2 K_2\sqrt{(L_1 K_1 - L_2 K_2)^2 + 4M^2 K_1 K_2}] \end{aligned}}{2K_2[R_2 (L_1 K_1 - L_2 K_2)^2 + R_1 M^2 K_1^2]}$$

It will be noticed that R_1 and R_2 are involved in the same way in all the denominators, and that the numerators differ only by a constant factor which does not involve the resistances, except the first, which has a term in $\frac{R_2}{R_1}$. Solving these equations for $R_2(L_1 K_1 - L_2 K_2)^2 + R_1 M^2 K_1^2$ and dividing by $M^2 K_1^2$

$$(28)\quad R_1 + R_2\frac{(L_1K_1 - L_2K_2)^2}{M^2K_1^2} = \frac{nV_0^2\left(L_2 + L_1\frac{R_2}{R_1}\right)}{V_2^2} = \frac{nK_2V_0^2}{V_2^2}$$
$$= \frac{nC_1^2[(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2 - (L_1K_1 - L_2K_2)\sqrt{(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2}]}{2M^2K_1^2K_2I_1^2}.$$

In the case where the secondary circuit is closed, the expression for the current is of the form

$$(29)\quad I = \frac{dQ}{dt} = e^{-\alpha t}[(-\alpha A + \beta B)\cos\beta t + (-\beta A - \alpha B)\sin\beta t] - \gamma Ce^{-\gamma t}.$$

The integral $\int_0^\infty I^2dt$ then consists of two principal parts. The last is, by direct integration, $\frac{\gamma^2C^2}{2\gamma} = \frac{\gamma C^2}{2}$. The first part, by the preceding discussion, is

$$\frac{(-\alpha A + \beta B) + (-\beta A - \alpha B)^2}{4\alpha} = \frac{(\alpha^2 + \beta^2)(A^2 + B^2)}{4\alpha}.$$

Then in the primary circuit

$$\int_0^\infty I_1^2dt = \frac{(\alpha^2 + \beta^2)(A_1^2 + B_1^2)}{4\alpha} + \frac{\gamma C_1^2}{2}.$$

By making the proper substitutions and disregarding small quantities this may be reduced to the form

$$(30)\quad \int_0^\infty I_1^2dt = \frac{V_0^2L_2^2K}{2(L_2^2R_1 + M^2R_2)}.$$

In the secondary circuit

$$\int_0^\infty I_2^2dt = \frac{(\alpha^2 + \beta^2)(A_2^2 + B_2^2)}{4\alpha} + \frac{\gamma C_2^2}{2}$$

which similarly can be reduced to the form

$$(31)\quad \int_0^\infty I_2^2dt = \frac{V_0^2M^2K}{2(L_2^2R_1 + M^2R_2)}.$$

Description of Apparatus.

In the experiments to be described the immediate source of current was a large induction coil, capable of giving at the secondary terminals on open circuit an effective difference of potential of twenty-one thousand volts, when operated from the commercial alternating circuit of fifty volts. This was excited in various ways, by current from a storage battery, by the commercial circuit spoken of above, and by current from a small alternator kindly loaned by Professor Pupin, of Columbia University.

The condensers in the primary circuit of the oscillating system were sheets of micanite, $10 \times 12 \times 1/40$ inches, coated on both sides with tin-foil to within about an inch and a-half of the edge. They were arranged symmetrically in two groups of two, and their capacity measured in electromagnetic units by the method suggested by Maxwell[1] and employed by J. J. Thomson[2] and Glazebrook.[3]

The condenser employed in the secondary circuit consisted of two circular brass disks, slightly convex, of about ten centimeters diameter, immersed in kerosene oil (petroleum). Its capacity was computed, approximately, but no attempt was made to measure it.

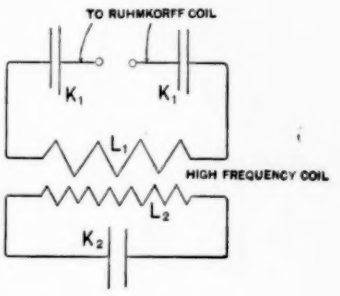

Fig. 1.

The primary coil contained 34.5 turns of heavy wire, was 22 cms. long and 8.3 cms. in mean diameter. The secondary had 84 turns in three layers, was about 30 cms. long and 10.6 cms. in external diameter. The coefficients of induction were measured by the simple bridge method suggested by Maxwell,[4] using alternating current and telephone, and as a standard a coil of rectangular cross section, whose self induction was computed by the method of Stefan.[5]

The primary spark took place between two balls of zinc 2 cms. in diameter, and was blown out by an air blast from a Sturtevant

[1] Treatise, Vol. II., § 776.
[2] Phil. Trans. 174, part 3, p. 707, 1883.
[3] Phil. Mag. (5), 18, p. 98, 1884.
[4] Treatise, Vol. II., §§ 756, 757.
[5] Wied. Ann. 22, pp. 107–117, 1884.

blower, driven by a small electric motor. The phenomena so obtained were more regular than when the spark occurred in oil.

The electrical dimensions of different parts of the system in c. g. s. absolute electromagnetic units are as follows:

$1{,}105{,}000$ = Self induction of standard coil.
$L_1 = 54{,}000$ = Self induction of primary coil.
$L_2 = 454{,}000$ = Self induction of secondary coil.
$M = 77{,}000$ = Mutual induction of the two coils.
$K_1 = 1.6 \times 10^{-18}$ = Capacity of primary condenser.
$K_2 = 2 \times 10^{-20}$ = Capacity of secondary condenser when present.

The resistances of the two coils to steady currents are small of the order of .05 and .3 ohms respectively. R_1 and R_2 will, however, contain not only these, increased perhaps considerably on account of the peripheral distribution of the current, but also the resistance of whatever measuring instruments are inserted, and of the spark gaps, where such exist.

PERIOD.

If in equation (19) we insert these values, we we find

$$A_1 = .03 V_0 K_1, \qquad C_1 = .97 V_0 K_1.$$

That is, the oscillation whose period is determined by the value of δ decidedly predominates in the primary circuit. This is due simply to the choice of dimensions of the system. The corresponding frequency hardly differs from the natural frequency of the primary system.

The experimental determinations of the period of oscillation were made by photographing a spark by means of a rotating mirror. The mirror itself was concave, silvered on the face, of about 36 cms. focal length, and mounted on the end of the shaft of an electric motor. The photographic plates were set at a distance of 81.5 cms. from the center of the face of the mirror, and the speed of the motor was determined by comparison with a standard tuning fork by a stroboscopic method.

The photographs of the most value were taken of the spark in the primary circuit. Some were taken also of that in the secondary

circuit, but these seem by the theory to represent an oscillation superposed upon a current dying away logarithmically, and the photographs are correspondingly hazy. In each photograph there appear several distinct sparks, each showing fine striations, which indicate the oscillations.

In Table I. are given a the number of revolutions per second of the mirror, b the number of oscillations distinctly visible in a given photograph, c the mean length of an oscillation, d the double frequency of oscillation, computed from a and c.

The dimensions of our apparatus would give, substituting in equation (9) $\delta = 3{,}400{,}000.$

and the frequency would be

$$\frac{\delta}{2\pi} = 542{,}000.$$

TABLE I.

a	b	c	d
50	4	.05	1.026×10^6
50	6	.05	1.026
50	9	.053	.963
50	4	.05	1.026
50	6	.053	.963
50	7	.051	.998
50	6	.047	1.097
50	10	.05	1.026
50	6	.05	1.026
64	4	.06	1.093
64	3	.063	1.037
64	5	.07	.939
64	7	.064	1.026
64	5	.068	.967
64	7	.063	1.044
		Average	1.017×10^6

The mean observed double frequency, from the table, is 1,017,000 which would give the observed frequency 510,000 nearly. This is as good a degree of agreement as could be expected considering the degree of accuracy of our knowledge of the constants of the system.

MAXIMUM POTENTIAL.

It appears from equation (20) that the greatest difference of potential which we can have in the secondary circuit is

$$\frac{2C_2}{K_2} = \frac{2V_0MK_1}{\sqrt{(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2}} = 2.7V_0.$$

The maximum potential was tested, roughly, by the measurement of spark-lengths, using for potentials of less than 30,000 volts determinations made by myself with the alternating current, upon the absolute electrometer, higher potentials were taken from curves

TABLE II.

a	*b*	*c*	*d*	*e*
.8	24,500	3.19	53,100	2.165
"	"	2.84	51,200	2.09
"	"	3.3	53,700	2.19
"	"	2.5	49,200	2.01
"	"	2.89	51,500	2.1
"	"	2.11	46,200	1.88
"	"	2.25	47,300	1.93
"	"	2.07	45,900	1.87
"	"	2.51	49,200	2.01
"	"	2.4	48,400	1.97
.6	19,450	1.87	44,200	2.27
"	"	2.65	50,400	2.59
"	"	1.5	40,400	2.08
"	"	1.28	37,200	1.91
"	"	1.06	33,000	1.7
"	"	1.35	38,400	1.97
"	"	1.24	36,600	1.88
.4	13,650	1.08	30,400	2.225
"	"	.75	23,500	1.72
"	"	.68	21,650	1.585
"	"	.775	24,100	1.76
"	"	.78	24,200	1.77
"	"	.725	22,800	1.67
"	"	.75	23,500	1.72
"	"	.765	23,900	1.75
.2	7,300	.555	18,300	2.505
"	"	.27	9,500	1.30
"	"	.325	11,300	1.55
"	"	.475	16,000	2.19
"	"	.375	12,900	1.77

drawn from data given by Heydweiller,[1] potentials above 50,000 volts being obtained, when necessary, by extrapolation.

In the accompanying Table II., which gives a few out of a great number of determinations, column a gives the length of the primary spark, b is the corresponding potential, V_0, c is the length of the secondary spark, d the potential corresponding thereto, and e the ratio d/b, which should have for its upper limit, as shown above, the value 2.7. The extreme values found range from 1.3 to 2.74, with averages in different groups of from 1.7 to 2.34.

The measurements of the effective difference of potential in the secondary circuit were made by means of a modified quadrant electrometer used idiostatically. Only one of the quadrants was retained, and the needle was supported on a horizontal axis with jewelled bearings. These bearings were carried on glass pillars, but on account of the high frequency the metallic parts had to be electrically connected to the needle. Neglect of this precaution resulted in the destruction of one of the jewels. The needle had suitable adjustments for level and sensitiveness and carried a plane mirror, enabling its deflections to be read with mirror and scale. The whole was immersed in kerosene oil, to prevent sparking. The oil served also as a damper to mechanical motions, and to increase the sensitiveness. The instrument gave a calibration curve which was an almost perfect parabola. Its constant was frequently redetermined by the absolute attracted disk electrometer belonging to the University.[2] The effective currents were measured by a form of hot-wire ammeter or dynamometer due to Hertz.[3] The current traversed a fine German silver wire which held a small steel wire in equilibrium against the torsion of a spring. The heat due to the current expanded the wire and allowed the steel wire to rotate under the influence of the spring. The deflections were read with mirror and scale. These instruments were repeatedly calibrated, using a storage battery and known resistances,

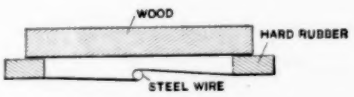

Fig. 2.

[1] Wied. Ann. 48, p. 213, 1893.
[2] See Edmondson, PHYSICAL REVIEW, Feb., 1898.
[3] Zeitsch. fur Inst. III., pp. 17–19, 1883. Ges. Werke, Bd. I., p. 227.

or current from a step-down transformer through a known non-inductive resistance, or by comparison with various Weston ammeters. The results were gratifyingly uniform. The sizes of wire used were Nos. 30, 36 and 40, with carrying capacity varying from 2 to .5 amperes. The instruments were very dead-beat, and particularly in the case of the smaller wires came to the final readings very promptly, and returned to zero almost as promptly.

The sensitive quadrant electrometer just described was connected in parallel with the secondary capacity K_2, and the two dynamometers were inserted in convenient positions in the primary and secondary circuits. After many trials of different positions, the dynamometer for the primary circuit was placed in the branch containing the spark gap.

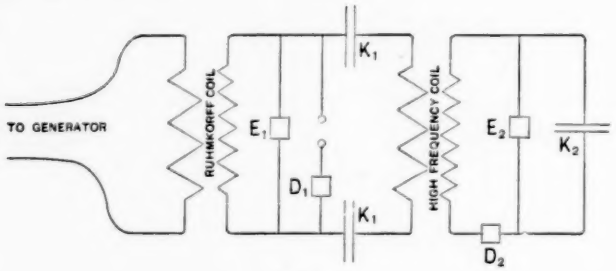

Fig. 3.

The terminals of the secondary circuit of the large induction coil were permanently connected to the absolute electrometer, as well as to the primary condenser of the oscillatory system.

In taking a series of observations, the primary spark gap is at first disconnected, and the current through the primary circuit of the Ruhmkorff coil is adjusted by inserting resistance, or varying the excitation of the dynamo. Then the terminal difference of potential of the primary condenser is determined by the absolute electrometer and recorded.

The next step is to connect in the primary spark gap, adjusting its length, if necessary. Then starting the blower, and allowing the spark to pass, readings are made of the deflections of the sensitive electrometer in the secondary circuit, and of both dynamometers. These readings are repeated several times, allowing the instruments

to return to zero between each reading, and then the primary spark gap is again removed, and the potential given by the Ruhmkorff again noted, for a check. The great variations of potential and frequency of the commercial circuit necessitated the use of an independent generator of current. Table III. contains part of the data thus taken. In column a is recorded the primary spark-length in centimeters, under V_0 the potential corresponding thereto, under b the maximum potential impressed upon the primary condenser when the spark gap is removed, computed on the assumption of a true sine-current. The columns $\overline{I}_1$, $\overline{I}_2$, $\overline{V}_2$, give the observed effective currents in both circuits, and potential in the secondary circuit, respectively.

The maximum impressed difference of potential b has been used as the most available parameter for the intercomparison of data, and is taken as abscissa in the accompanying plot, Fig. 6, which gives

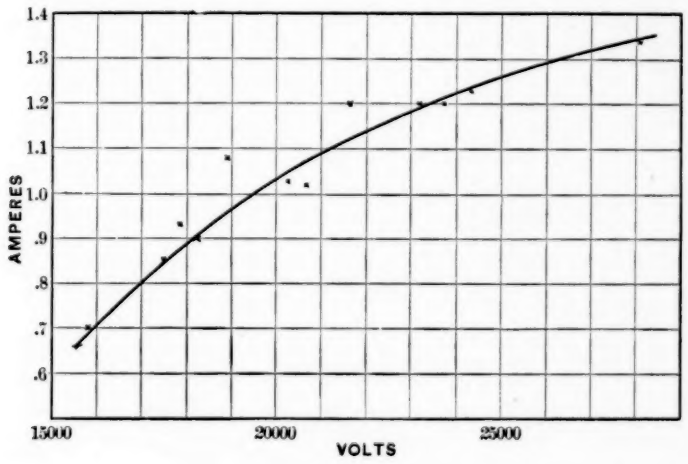

Fig. 4.

the observed effective primary current for a primary spark-length of 4 mm. All the curves for $\overline{I}_1$, $\overline{I}_2$, $\overline{V}_2$ are of similar character, and show a decided rise with what may be called increasing excitation, The same was true, but in less degree, of the maximum spark length in the secondary circuit, the data for which, in Table I.. however, are not classified with reference to this point. The ques-

tion immediately arises as to the reason for this behavior. The most obvious suggestion is, that on account of the excess of current supplied to the condenser the maximum potential effective at the primary spark gap is greater than that indicated by its length. This suggestion is decidedly negatived, however, by the fact that the spark length in the secondary circuit consistently falls short of the value possible on theoretical grounds. It would appear, rather, that the cause of the variation in our phenomena is the variable resistance of the primary spark, and that the helpful influence of increasing excitation is simply due to the increase of current poured through the spark gap at instants of formation of the spark, which serves to decrease its resistance.

If we substitute in equations (28) the values of the constants of

TABLE III.

SERIES I. $n = 125$.

a	V_0	b	$\overline{V}_2$	R_1
.2	7,300	9,900	188	85.5
"	"	11,410	260	45.
"	"	14,000	353	24.4
"	"	16,150	482	13.
"	"	18,050	606	8.25
.4	13,650	16,150	458	51.
"	"	19,800	651	25.
"	"	22,850	956	11.6
"	"	22,850	910	12.8
"	"	25,550	1,013	8.5
.6	19,450	22,120	511	82.4
"	"	22,800	723	41.1
"	"	24,900	871	28.3
"	"	25,550	825	31.6
"	"	28,600	970	22.8
"	"	29,950	1,210	14.7
"	"	27,400	1,190	15.2
"	"	30,250	1,230	14.2
.8	24,500	26,160	535	119.
"	"	26,160	671	72.
"	"	26,750	721	65.7
"	"	27,400	777	57.
"	"	27,700	921	40.2
"	"	28,550	975	36.
"	"	28,900	1,160	25.3

TABLE III.

SERIES II. $n = 136$.

a	V_0	b	$\overline{V_2}$	R_1
.2	7,300	8,670	186	95.4
"	"	9,030	208	70.1
"	"	11,850	232	61.2
"	"	14,240	302	36.1
"	"	18,070	350	27.0
.4	13,650	14,710	175	375.
"	"	16,650	290	136.
"	"	17,850	375	81.8
"	"	18,500	519	42.7
"	"	21,050	740	21.1
"	"	2,300	811	17.4
"	"	24,170	900	14.2
"	"	29,000	974	12.2
.6	19,450	19,770	204	560
"	"	20,600	207	543
"	"	21,770	402	144
"	"	23,450	492	96
"	"	25,200	710	46
"	"	26,800	950	25.9
"	"	28,000	975	24.6
.8	24,500	25,600	269	512
"	"	26,600	575	112
"	"	28,300	705	74
"	"	29,500	720	71.2
"	"	30,520	915	44

our system, we get, for $n = 125$,

$$R_1 + .387R_2 = \left(56.8 + \frac{6.75R_2}{R_1}\right) \times 10^6 \frac{V_0^2}{V_2^2}$$

$$= 2.5 \times 10^{-18} \frac{V_0^2}{\overline{I_2}^2}$$

$$= 190 \times 10^{-18} \frac{V_0^2}{\overline{I_1}^2},$$

and for $n = 136$

$$R_1 + .387R_2 = \left(61.8 + \frac{7.35R_2}{R_1}\right) \times 10^6 \frac{V_0^2}{V_2^2}$$

$$= 2.72 \times 10^{-18} \frac{V_0^2}{\overline{I_2}^2}$$

$$= 206 \times 10^{-18} \frac{V_0^2}{\overline{I_1}^2}.$$

TABLE III.

SERIES III. $n = 136$.

a	V_0	b	$\bar{I}_1$	R_1	$\bar{I}_2$	R_1
.2	7,300	9,900	.73	20.5	.038	100
"	"	10,400	.635	27.1	.037	106
"	"	14,100	1.04	10.1	.057	44.5
"	"	14,350	.98	11.4	.056	45.9
"	"	18,050	1.26	6.9	.07	29.6
"	"	18,500	1.19	7.7	.064	35.4
.4	13,650	15,650	.66	87.5	.038	352.
"	"	15,840	.70	77.8	.045	250.
"	"	18,250	.90	47.3	.058	150.
"	"	18,900	1.08	32.6	.065	120.
"	"	21,600	1.20	26.6	.075	90.
"	"	23,200	1.20	26.6	.072	98.
.6	19,450	20,200	.56	246.	.03	1,145.
"	"	21,400	.92	92.	.053	365
"	"	22,300	.97	82.	.058	307
"	"	26,500	1.30	46.3	.076	178
"	"	28,000	1.33	44.1	.075	182
"	"	29,100	1.33	44.1	.079	164
.8	24,500	25,850	1.03	116.	.056	521
"	"	26,060	.81	188.	.045	805
"	"	28,000	1.24	80.	.078	268
"	"	29,100	1.23	81.	.07	332
"	"	30,900	1.41	62.	.081	249
"	"	32,500	1.47	57.	.09	202

These values are for the absolute system of units. To change them to ohms, volts and amperes we must write for the exponents of $10 - 3, -9, -9$, respectively. The values of $R_1 + .387R_2$ computed according to these equations (assuming in the first that $\frac{R_2}{R_1}$ is small) are given in Table III. in the columns headed by R_1. R_2 is a purely metallic resistance, while R_1 contains the spark gap, so that the resistance of this spark is in all probability the greater part of the resistance $R_1 + .387R_2$.

The numerical values obtained from these different sources are by no means identical, but the results deduced from the values of $\bar{V}_2$ and $\bar{I}_1$ will be seen on inspection to agree quite fairly well, and all the results are concordant to this extent, that the values of the spark resistance, as thus given, are all of the same order, and

that this resistance is a variable, but not linear function of the current in the spark. Fig. 7 gives R_1 for the same spark gap as Fig. 6, namely, 4 mm., using the same abscissa.

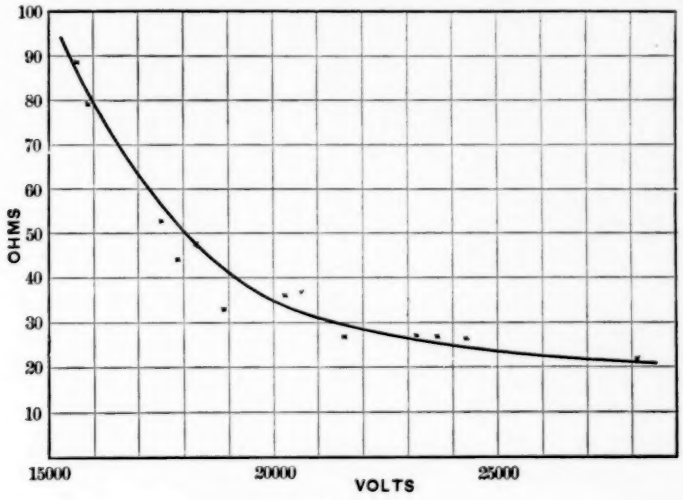

Fig. 5.

Whether this resistance falls off indefinitely, or approaches some finite limit cannot be told from the limited amount of data here presented.

Closed Secondary Circuit.

Substituting in equation (21) the values of the capacity and inductances of our system we get

$$\beta = 3.905 \times 10^6$$

which gives us the frequency

$$\frac{\beta}{2\pi} = 622000$$

No direct measurements were made verifying this frequency. The few spark photographs made show mainly the hazy light due to the current expressed by the exponential term.

Substituting the values of capacity and inductances in equations (30) and (31), and reducing from the absolute to the practical system we get

$$\int_0^\infty I_1^2 dt = V_0^2 \frac{8 \times 10^{-10}}{R_1 + .1694 R_2}$$

$$\int_0^\infty I_2^2 dt = V_0^2 \frac{1.355 \times 10^{-10}}{R_1 + .1694 R_2}$$

Solving for $R_1 + .1694 R_2$ we get for $n = 1361$

$$R_1 + .1694 R_2 = 1.089 \times 10^{-7} \frac{V_0^2}{I_1^2}$$

$$= 1.845 \times 10^{-8} \frac{V_0^2}{I_2^2}$$

Table IV. gives in columns *a*, *V*, *b*, as before the primary spark length, the potential corresponding thereto, and the maximum impressed potential. In column *c* are given the lengths of spark gaps introduced into the secondary circuit, the spark taking place between brass balls 2 cms. in diameter. Columns $\bar{I}_1$ and $\bar{I}_2$ give the observed effective currents in the two circuits, while columns *d* and *e* give $R_1 + .1694\, R_2$ computed from $\bar{I}_1$ and $\bar{I}_2$ respectively by the equations just given.

It will be seen by reference to the table that the values of the resistance here found are of the same order as those found in the case of open secondary circuit.

It has been mentioned that the resistances R_1 and R_2 consist both of spark gap and of metallic resistance.

Gray and Mathews[1] show that the virtual resistance of a straight metallic wire to very rapidly oscillating currents is

$$R' = \sqrt{\frac{\mu n l R}{2}}$$

Taking μ as unity, this can be reduced to the form

$$R' = Rr \sqrt{\frac{\pi n}{2k}}$$

where k is the conductivity. For $n = 500{,}000$ and $k = .0006$ this gives the rather startling result

$$R' = 36{,}000 rR$$

which for wire of 1 mm. diameter would be

$$R' = 1{,}800 R$$

[1] Treatise on Bessel's Functions, p. 160.

TABLE IV.

a	V_0	b	c	$\overline{I}_1$	d	$\overline{I}_2$	e
.2	7,300	8,200	0	.136	315.	.067	130.
"	"	"	.2	.124	379.	.18	30.5
"	"	"	.3	.11	482.	.188	27.9
"	"	10,000	0	.83	8.5	.15	43.8
"	"	"	.1	.78	9.6	.2	24.7
"	"	"	.2	.73	10.95	.31	10.25
"	"	"	.3	.69	12.25	.38	6.84
"	"	11,430	0	.96	6.33	.168	35.
"	"	"	.1	.88	7.53	.225	19.5
"	"	"	.2	.85	8.09	.295	11.33
"	"	"	.3	.80	9.12	.415	5.73
"	"	12,800	0	1.06	5.2	.163	37.1
"	"	"	.1	1.02	5.6	.295	11.33
"	"	"	.2	.97	6.2	.355	7.82
"	"	"	.3	.95	6.47	.438	5.13
"	"	17,300	0	1.25	3.7	.215	21.3
"	"	"	.1	1.17	4.26	.367	7.33
"	"	"	.2	1.14	4.48	.41	5.88
"	"	"	.3	1.11	4.73	.49	4.11
.4	13,660	14,900	0	.7	41.5	.128	210.
"	"	"	.1	.66	46.7	.237	61.3
"	"	"	.2	.61	54.7	.405	21.
"	"	"	.3	.54	69.8	.5	13.8
"	"	18,100	0	1.15	15.33	.16	134.5
"	"	"	.1	1.12	16.2	.244	58.
"	"	"	.2	1.08	18.1	.311	35.6
"	"	"	.3	1.00	20.3	.481	14.9

This deduction assumes, however, that the wire is at an infinite distance from other currents, while in our case the distance between wires is comparable to their diameters. The results of our experimental work would also entirely contradict any assumption of such excessive increase in metallic resistance.

A brief comment upon the degree of accuracy attained and attainable in such work may be of interest. The behavior of the dynamometers left nothing to be desired. They acted with much greater uniformity than the phenomena to be observed, so that any irregularity observed in their readings must be attributed to actual variations in the currents. As much can hardly be said of the electrometer. To give convenient readable deflections with the

mean potentials observed it required to be adjusted with such sensitiveness that the directive force was not large enough to prevent frictional disturbance of the position of equilibrium. Further, the inertia of the moving system was such as to prevent prompt reading of deflections, and in case of intermittent action the readings obtained were a time average, which was necessarily small. The observations were of great value, however, because they were of a wholly different type of phenomenon, and furnished so good a check upon both the theoretical reasonings and the accuracy of the other work. In general the accuracy of the results obtained seems to have been conditioned almost entirely upon the uniformity of the phenomena of a blown-out spark in air.

In the foregoing work an attempt has been made to verify experimentally the agreement of the actual behavior of an oscillating system with two degrees of freedom with the approximate theory. As specific conclusions resulting from this comparison we see that :

1. The main period of oscillation of the primary circuit is very nearly that deduced from the dimensions of the system. The same may be said also of the maximum potential attained in the secondary circuit.

2. The effective currents and potentials, which are functions of the damping factors, and these in turn factors of the resistances, would indicate that the resistances of the sparks are of the order of from ten to one hundred ohms, depending upon the amount of current flowing through the spark. This conclusion is in gratifying agreement with the work of Trowbridge and Richards,[1] who have similarly used the damping effect upon an oscillatory current to measure the resistance, but have done this by direct substitution.

3. It appears from Table IV. that when the secondary circuit is closed by a spark, the primary current decreases with the length of this spark, but the secondary current decidedly increases. This behavior is not explained by the approximate theory here deduced, but was most unmistakeable, both in early preliminary work, and in the later more careful determinations here recorded. It still remains to be shown whether this is due to the conditions of the experiment,

[1] Phil. Mag. (5) 43, pp. 349–367, 1897.

or is to be explained by a more accurate application of theoretical reasoning.

In conclusion it only remains for me to express my thanks to Professor A. G. Webster for his unfailing sympathy and helpfulness, which has rendered this work possible, and to Clark University, which placed at my disposal the facilities for the work.

NEW BOOKS.

Canada's Metals. By W. C. ROBERTS-AUSTEN. 8vo, pp. 46. London: Macmillan & Co. 1898.

Those who were so fortunate as to hear Professor Roberts-Austen's lecture on Canada's Metals at the Toronto meeting of the British Association will hail its appearance in print with pleasure. The little volume is well printed and fully illustrated. With the references and footnotes that have been added it will be found not only very interesting reading but also a valuable source of information on the subjects treated.

The lecture contains more of special interest to physicists than would appear from its title. The first half of the book, being devoted to an account of the mining industries of the Dominion and to the history of their development, appeals perhaps most directly to the geologist or to the general reader; but having told us what ores occur and where they are to be found, the lecturer next proceeds to discuss the physical properties of the metals themselves. This part of the lecture, which was illustrated at Toronto by numerous experiments, forms the second half of the book.

One of the most striking features of the book is the series of excellent reproductions of photographs illustrating the "flow" of solid metals. The first photographs, due to Worthington, give instantaneous views of the splash produced by a marble dropping into milk. A similar series taken by the lecturer shows the splash produced by a bullet falling into molten gold. Finally, we are shown the "splash" produced in an armor plate by the impact of an armor-piercing projectile. Certainly, after seeing these photographs, no one can deny that the lecturer has proven his point, and that solid steel may sometimes behave like a viscous liquid.

The discussion of the properties of the nickel-steel alloys; the experiments upon the melting point and heat of fusion of gold; and the arrangement of an electric furnace for projection, are all points which I can mention merely. All who have an interest in physical science will find pleasure in reading about "Canada's Metals."

E. M.